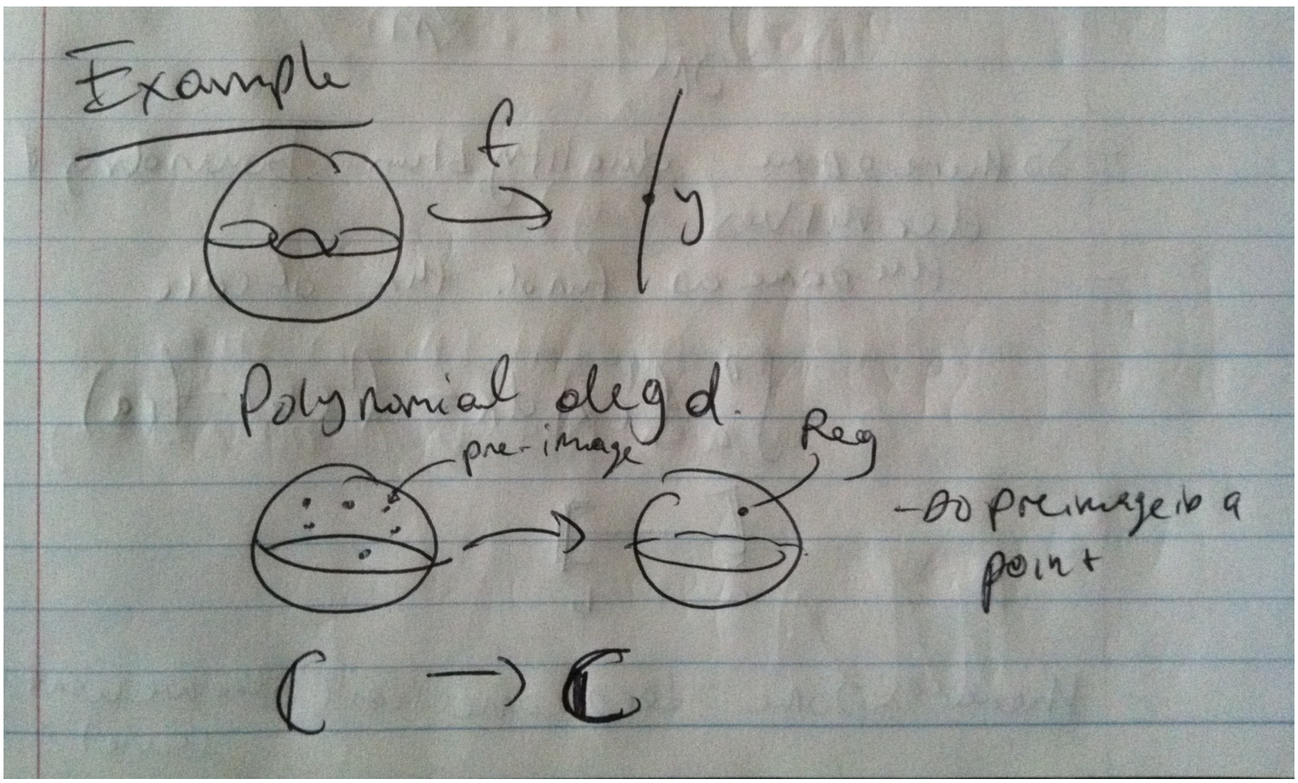
Notes from October 7th:

Last time:

We proved this very important lemma:

If $f: M \rightarrow N$,where M is m dimensional and N is N dimensional, is smooth. Then if $y \in N$, is a regular value then $f^{-1}(y)$is a smooth manifold of dimension (m-n).

Example:



The idea is cuz $Kernal(df_x)$ is a subspace of TM_x

Now:

What if we have a manifold with boundary?

-We denote the boundary of M with ∂M
-This choice of notation is reasonable because there is some connection between the derivative and the boundary. This is made most clear in Stoke's theorem.

$$\int_{\partial M} f(x) = \int_M \partial f(x)$$

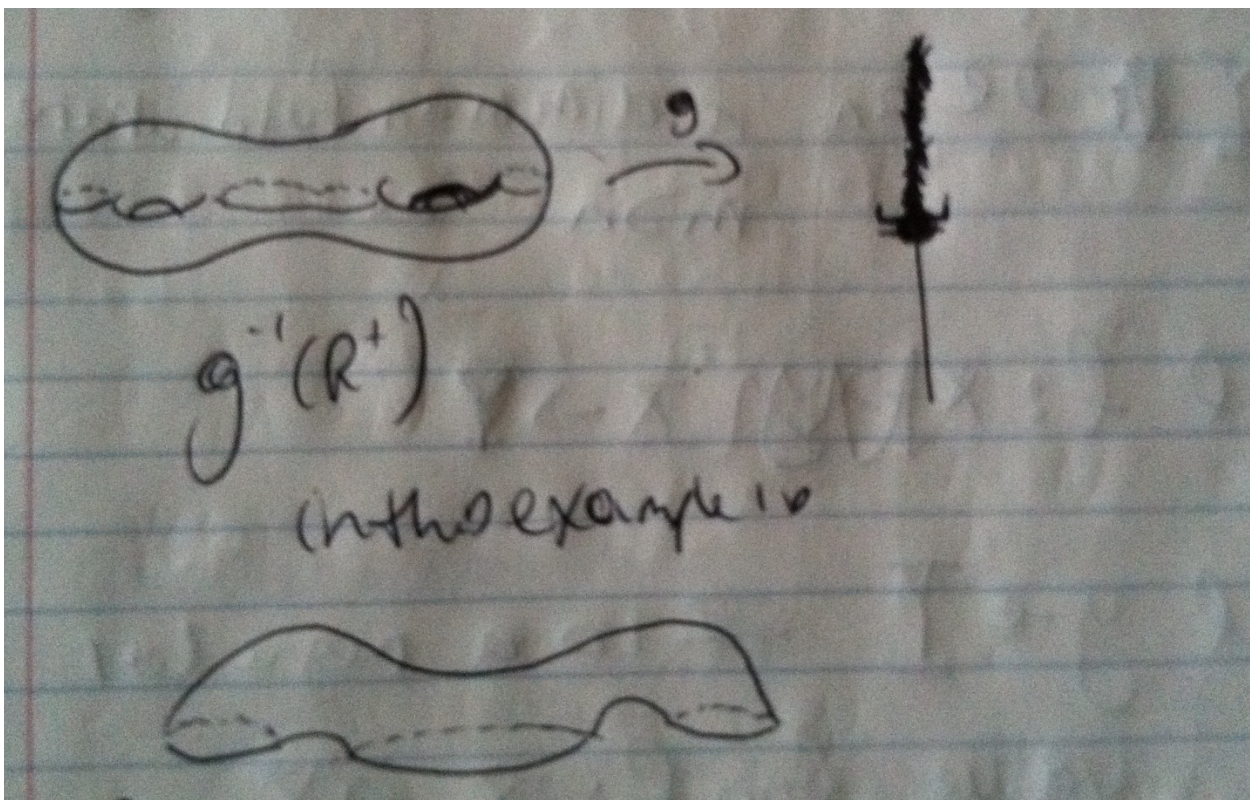
-This is the same idea as the fundamental theorem of calculus from one dimensional calculus where

$$\int_a^b f'(x) = f(b) - f(a)$$

-So now let Mbe a manifold without boundary and let $g: M \to \mathbb{R}$. What will $\{x \in M | g(x) \geq 0\}$ be?
-Answer: A manifold with boundary. Now let's show this.

<u>Sketch of Proof with Diagram:</u>

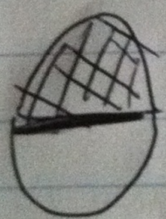
$$g: M \to \mathbb{R}$$



So g maps from an m dimensional manifold to a 1 dimensional manifold. Also The inverse exists at $g(0)$.
We have established therefore by the lemma from last time that $g^{-1}(0)$ is a smooth (m-1) dimensional sub manifold of M.

$g^{-1}(0)$ is a smooth m-1 dim

sub ~~M~~ mani fold of M.



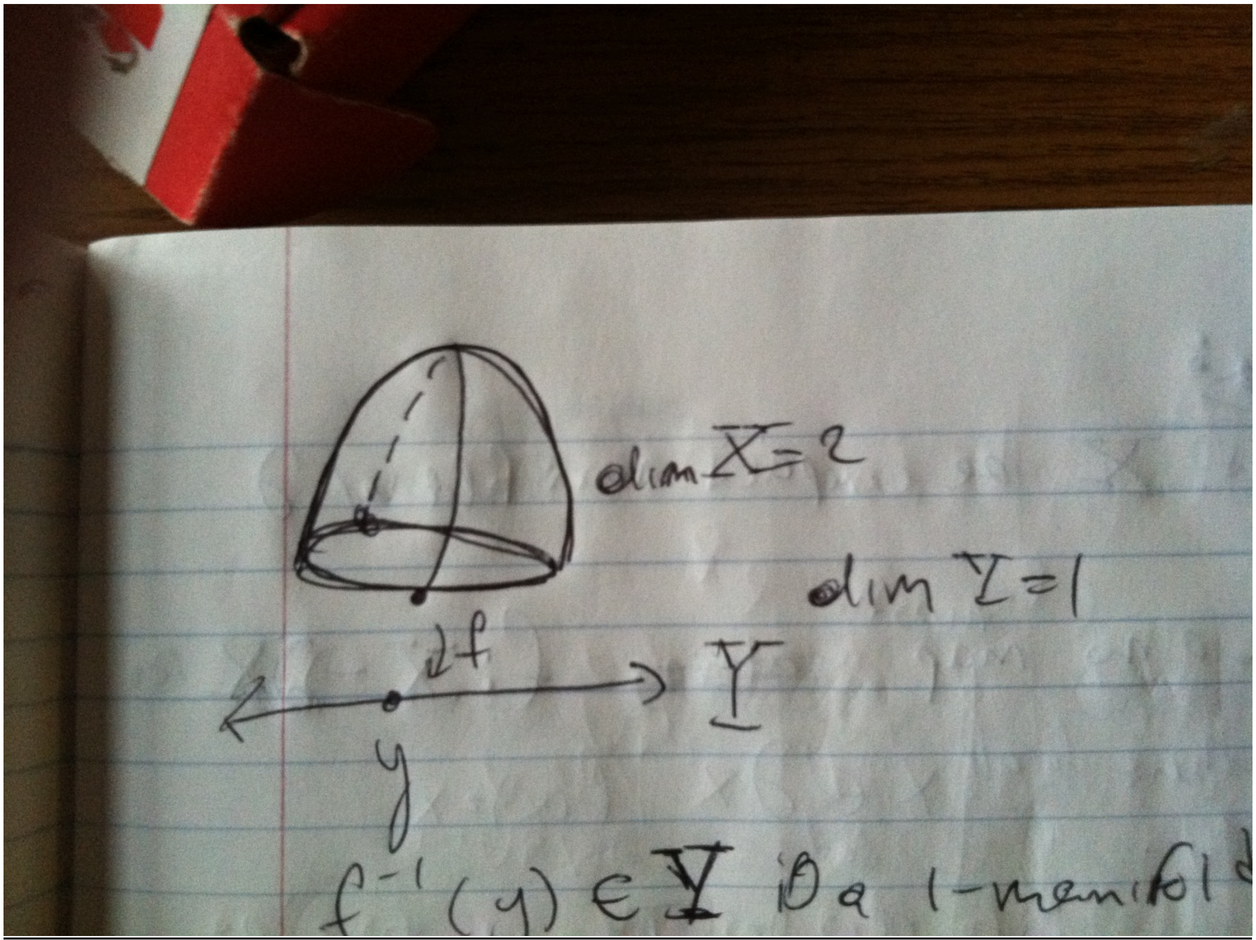
So if we limit the domain of inverse to $\mathbb{R}^+$, then it is easy to se that the manifold $g^{-1}(0)$ will become the boundary.

-This is a really easy way to generate a manifold with boundary.
-Let’s state another really vital Lemma:

Let X be an m-dimensional manifold with ∂. Let Y be an n-dimensional manifold where m>n. Also let $f: X \to Y$

If $y \in Y$ is a regular value for $f: X \to Y$ and $f|_{\partial X}: \partial X \to Y$ then $f^{-1}(y)$ is a smooth (m-n) sub-manifold with boundary.

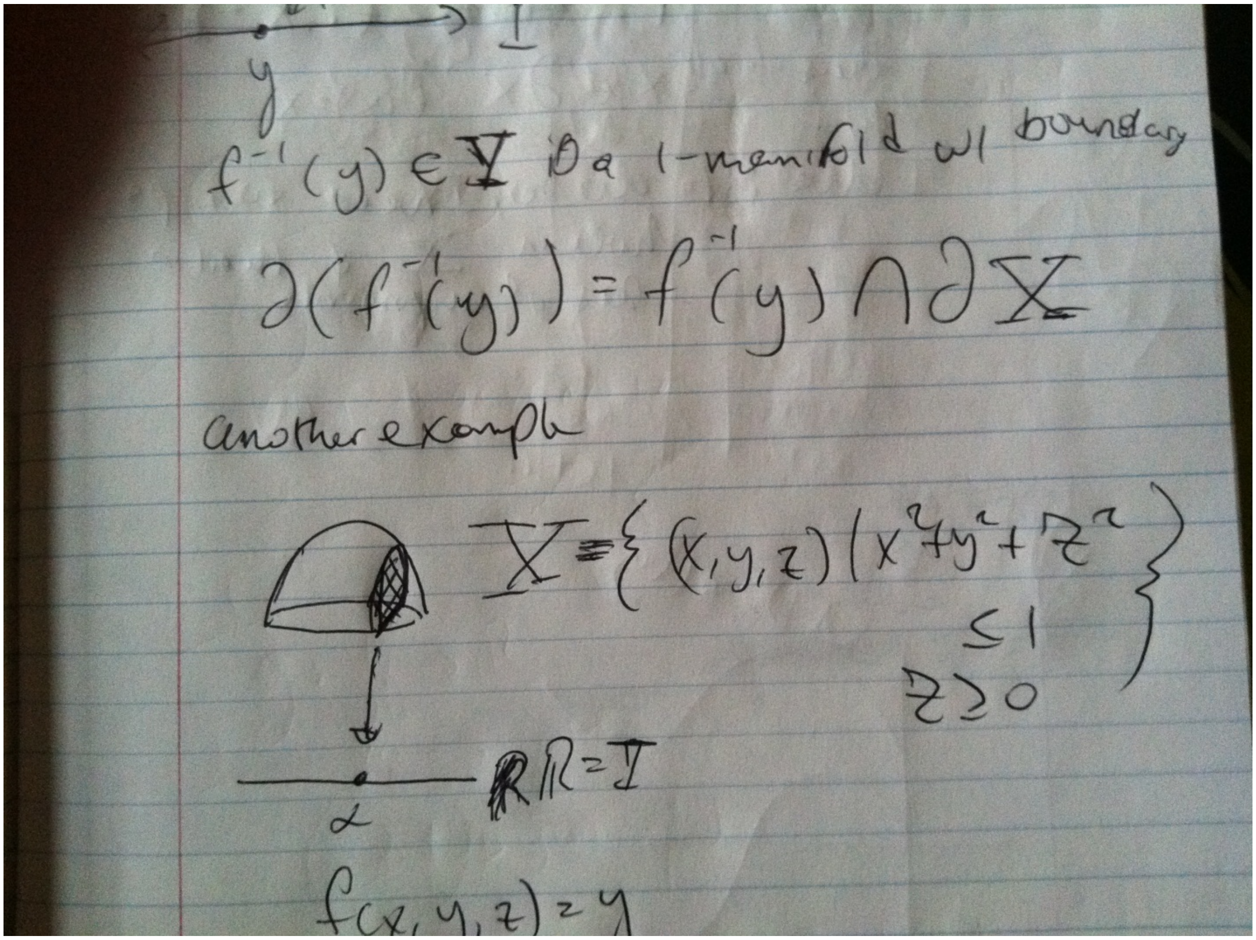
<u>Example:</u>



$f^{-1}(y \in Y)$ is a 1-dimensioal manifold with boundry.

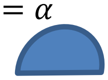
$$\partial(f^{-1}(y)) = f^{-1}(y) \bigcap \partial X$$

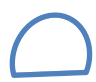
Another example:



$$X = \{(x, y, z) \ / \ x^2 + y^2 + z^2 \leq 1 \ \& \ z \geq 1\}$$
$$Y = \mathbb{R}$$

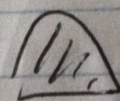
$$f(x, y, z) = \alpha$$

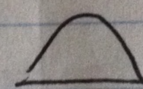
$f^{-1}(\alpha) =$ 

$\partial(f^{-1}(\alpha)) =$ 

$$= \partial(f^{-1}(\alpha)) \bigcap \partial X$$

$$f(x, y, z) = y$$

$$f^{-1}(\alpha) =$$ 

$$\partial\left(f^{-1}(\alpha)\right) =$$ 

$$= f^{-1}(\alpha) \cap \partial X$$

<u>Lemma:</u>

Let X be a compact manifold with ∂.

Then there is no smooth map $f\colon X \to \partial X$ so that $\forall\, x \in \partial X$, $f(x) = x$

Idea: to create such a function, I need to rip a hole in X and that is sort of impossible.