Notes for MAT 364, 9/21/2011

Chain Rule:

Suppose we have $f : U \to V$, $g : V \to W$, where $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^n$, and $W \subset \mathbb{R}^k$. If f and g are *smooth* at x and y respectively, then $d(g \circ f)_x = dg_y \circ df_x$.

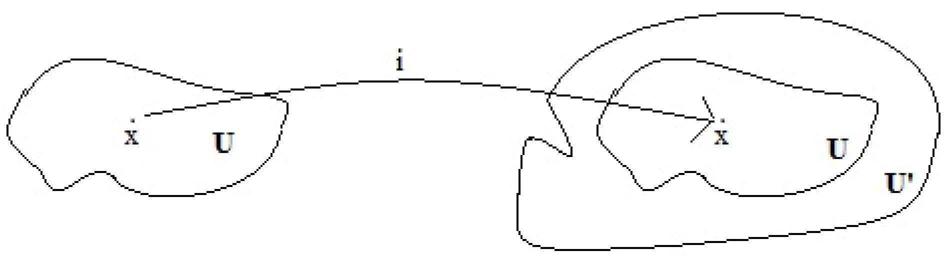
Here is a diagram on how the sets and functions interact:

$$\begin{tikzcd} & V \arrow[dr, "g"] & \\ U \arrow[ur, "f"] \arrow[rr, "g\circ f"] & & W \end{tikzcd} \quad \overset{f,g \text{ are smooth at } x,y}{\Longrightarrow} \quad \begin{tikzcd} & \mathbb{R}^m \arrow[dr, "dg_y"] & \\ \mathbb{R}^n \arrow[ur, "df_x"] \arrow[rr, "d(g\circ f)_x"] & & \mathbb{R}^k \end{tikzcd}$$

Identity and inclusion maps:

If $I : U \to U$ is the *identity map* ($I(x) = x$, $\forall x \in U$) with $U \subset \mathbb{R}^n$, then dI is the *identity map* of $\mathbb{R}^n$.

More generally, if $i : U \to U^{'}$ is an *inclusion map* ($i(x) = x$, $\forall x \in U$) with $U \subset U^{'} \subset \mathbb{R}^n$, then $i|_U$ is the *identity map* of U, i is not defined on $U^{'} - U$, and di_x is the *identity map* of $\mathbb{R}^n$.



For example: Suppose $U = \{(x,y)|x^2 + y^2 < 1\}$ (unit disc), and $U^{'} = \{(x,y)|x^2 + y^2 < 4\}$. Consider $i : U \to U^{'}$ such that $i(x,y) = (x,y)$ for all $(x,y) \in U$. Then $di_{(x,y)}$ is the identity map on $\mathbb{R}^2$. The tangent space of U, $T_{(x,y)}U = \mathbb{R}^2$ with $di_{(x,y)} : \mathbb{R}^2 \to \mathbb{R}^2$.

Linear map, diffeomorphism:

If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a *linear map*, then $dL_x : \mathbb{R}^n \to \mathbb{R}^m$ is L. (derivative is a linear approximation of L.)

Corollary: Suppose that $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$. If $f : U \to V$ is a *diffeomorphism* at x, then $df : \mathbb{R}^n \to \mathbb{R}^m$, $n = m$ and df_x is invertible.

Reminder: Given two manifolds M, N. $f : M \to N$ is a *diffeomorphism* if and only if f is *bijective* and f and f^{-1} are *differentiable*.

Proof for Corollary:

Since f is a *diffeomorphism*, $f^{-1} : V \to U$ is a *differmorphism* and $(f \circ f^{-1})$, $(f^{-1} \circ f)$ are *identity maps* of V and U respectively.

Thus, $d(f \circ f^{-1})$ is identity on $\mathbb{R}^m$ and $d(f^{-1} \circ f)$ is identity on $\mathbb{R}^n$. Since $df \circ df^{-1}$ is identity, both df and df^{-1} has full rank (or empty kernel). Hence, $n \geq m \geq n$. Thus, $n = m$.

Converse: The converse is only true locally.

If $f : U \to V$, and $\exists x \in U$ such that df_x is non-singular and $n = m$, then f is *diffeomorphism* locally.

For example: $f(x) = \sin(x)$ is not invertible over the whole $\mathbb{R}$ but is invertible between any 2 critical points.

Definition (Critical point): If df_x is singular, then x is a critical point of f, otherwise, x is a regular point of f.
Aside: Idea of critical point does not need derivative. Consider the function $f(r, \theta) = (r, 2\theta)$ in polar coordinate which maps a half circle to full circle. $(r, 0)$ is a critical point.

Goal (not finished): Fundamental Theorem of Algebra

If $P : \mathbb{C} \to \mathbb{C}$ is a polynomial of degree d, $P(z) = 0$ has exactly d solutions with multiplicity.

Recall that $\mathbb{C} \leftrightarrow \mathbb{R}^2$ by $(a + ib) \to (a, b)$.

Addition in $\mathbb{C}$ works the same as $\mathbb{R}^2$

We want to extend the vector space Structure of $\mathbb{R}^2$ by allowing multiplication using $i^2 = -1$.

Consider $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ = Riemann sphere via Stereographic projection.

Recall: Stereographic projection

Consider $\mathbb{C}$ as a plane and S^2 as a unit sphere. Pick out the north pole $(0, 0, 1)$ and draw a line connecting a point z on $\mathbb{C}$ and $(0, 0, 1)$. The projection of z is point Z where it is the interception of S^2 and the line.

Here is a picture that wikipedia has: (I can't draw 3D)