MAT364, Homework 4

due wednesday 10/5

1. Recall that a set A is *connected* if it cannot be written as the union of two nonempty, disjoint open sets (here "open" means open relative to A; if you prefer, you could say that it is not possible to find two disjoint open sets U_1, U_2 in $\mathbb{R}^n$ such that $A \cup U_i \neq 0$ and $A \subset (U_1 \cup U_2)$. These are the same thing). A set B is *path-connected* if for any two points x and y in B, there is a path $\gamma \in B$ which connects x to y.

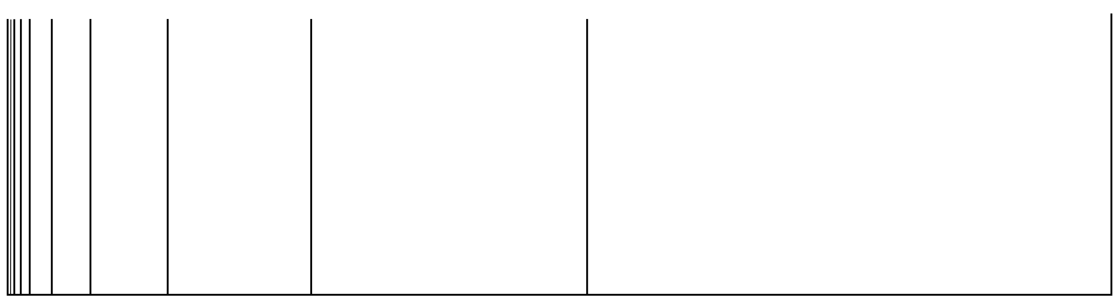
   a. Prove that any path-connected set is also connected.

   b. Suppose that A is a connected set.
   Is the interior of A necessarily connected? (prove or give a counterexample)
   Is the closure of A necessarily connected? (prove or give a counterexample)
   Is the closure of A necessarily path-connected? (prove or give a counterexample).

   c. Suppose M is a connected manifold. Prove that M is path-connected.

2. A set is *locally connected at x* if every neighborhood of x contains a connected open neighborhood.
   Consider the comb space $C \subset \mathbb{R}^2$:

$$C = \bigcup_{n=1}^{\infty} \left\{ \left(\frac{1}{n}, y \right) \mid 0 \leq y \leq 1 \right\} \cup \{(x, 0) \mid 0 \leq x \leq 1\} \cup \{(0, y) \mid 0 \leq y \leq 1\}$$



Show that C is path-connected but not locally connected.