MAT360 Midterm Exam

March 26, 2009

Name: ______________________ **ID:** ________________

Question:	1	2	3	4	5	6	Total
Points:	10	10	10	10	10	0	50
Score:							

There are 6 problems in this exam. Make sure that you have them all.

Do all of your work in this exam booklet, and cross out any work that the grader should ignore. You may use the backs of pages, but indicate what is where if you expect someone to look at it. **Books, calculators, extra papers, and discussions with friends are not permitted.** This is an "open book" exam only if the open book contains no math and is in another room. You may use any results obtained by consulting a talking fish, but must give the fish appropriate credit when doing so.

You have 80 minutes, more or less, to complete this exam.

This version has the typos fixed, unlike the one people took.

Geometric Axioms

UNDEFINED TERMS: *point, line, incident, between, congruent.*

INCIDENCE AXIOMS

I1: For any two distinct points P and Q, there is a unique line l incident with both of them.

I2: For any line l, there are at least two distinct points P and Q incident with l.

I3: There are at least three distinct, non-collinear points.

BETWEENNESS AXIOMS

B1: If a point B lies between points A and C, (written $A \star B \star C$), then A, B, and C are distinct collinear points, and $C \star B \star A$.

B2: Given any two distinct points B and D, there exist points A, C, and E lying on $\overleftrightarrow{BD}$ so that $A \star B \star D$, $B \star C \star D$, and $B \star D \star E$.

B3: If A, B, and C are three distinct points on the same line, then exactly one of the points is between the other two.

B4: **(Plane Separation)** For every line l and any three points A, B, and C not lying on l:

(a) If A and B are on the same side of l and B and C are also on the same side of l, then A and C are on the same side of l.

(b) If A and B are on opposite sides of l and B and C are also on opposite sides of l, then A and C are on the same side of l.

CONGRUENCE AXIOMS

C1: If A and B are distinct points and if A' is any point, then for each ray r emanating from A' there is a unique point B' on r such that $B' \neq A'$ and $\overline{AB} \cong \overline{A'B'}$

C2: If $\overline{AB} \cong \overline{CD}$ and $\overline{AB} \cong \overline{EF}$, then $\overline{CD} \cong \overline{EF}$. In addition, every segment is congruent to itself.

C3: If $A \star B \star C$, $A' \star B' \star C'$, $\overline{AB} \cong \overline{A'B'}$, and $\overline{BC} \cong \overline{B'C'}$, then $\overline{AC} \cong \overline{A'C'}$.

C4: Given any $\angle BAC$ and any ray $\overrightarrow{A'B'}$ emanating from a point A', then there is a unique ray $\overrightarrow{A'C'}$ on a given side of line $\overleftrightarrow{A'B'}$ such that $\angle B'A'C' \cong \angle BAC$.

C5: If $\angle A \cong \angle B$ and $\angle A \cong \angle C$, then $\angle B \cong \angle C$. Moreover, every angle is congruent to itself.

C6: **(SAS)** If two sides and the included angle of one triangle are congruent, respectively, to the two sides and the included angle of another triangle, then the two triangles are congruent.

CONTINUITY AXIOM

D: **(Dedekind's Axiom)** The points of any line are in bijection with the real numbers $\mathbb{R}$.

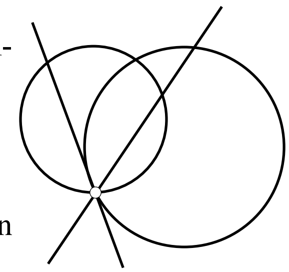
PARALLEL AXIOM

P: Given a line l and a point P not on the line, there is $\{$no, exactly one, more than one$\}$ line m through P which is parallel to l, depending on whether the geometry is $\{$elliptic, Euclidean, or hyperbolic$\}$. Note that we are not making a choice here, although elliptic geometry is inconsistent with the previous axioms. In this class, you should only use this axiom if you explicitly say so.

1. Consider the following interpretation of the terms "point", "line", and "incidence":
   - A point is any pair of real numbers (x, y) such that x and y are not both zero. That is, any point in $\mathbb{R}^2$ except the origin.
   - A line is the set of points (x, y) as above for which there are real numbers a and b not both zero such that either

     $$(x-a)^2 + (y-b)^2 = (a^2 + b^2) \qquad \text{or} \qquad ax = by.$$

     That is, a line is any circle or straight line passing through the origin in $\mathbb{R}^2$.



   - A point is incident with a line if it satisfies the corresponding equation.

   (a) $\boxed{\text{6 points}}$ Does this define an incidence geometry? That is, do axioms **I1**, **I2**, and **I3** hold? Fully justify your answer.[1]

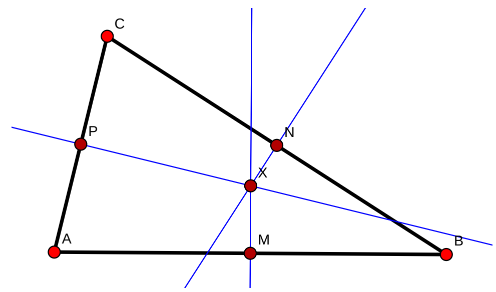
   (b) $\boxed{\text{4 points}}$ Does this model define a Euclidian, elliptic, or hyperbolic geometry (or none of the those)? Again, fully justify your answer.

[1]It might be useful for you to remember that for any three distinct non-collinear points in $\mathbb{R}^2$, there is a unique circle on which they all lie. Also, C.S. Lewis once said the following line: "The next best thing to being wise oneself is to live in a circle of those that are."

You may skip **one** of problems 2, 3, 4, 5, or 6 by crossing off the page.

2. $\boxed{\text{10 points}}$ Rewrite the following proof, supplying valid reasons for each of the steps.

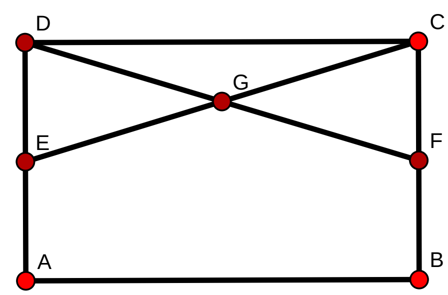
   In any triangle $\triangle ABC$, if the perpendicular bisectors of two of the sides of $\triangle ABC$ meet, all three perpendicular bisectors of the three sides meet in a common point X, called the *circumcenter*.

   Proof: Let M, N, and P be the midpoints of sides $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ respectively. Let the perpendicular bisector of $\overline{AB}$ meet that of $\overline{BC}$ at a point X. We must show that $\overline{PX}$ is perpendicular to $\overline{AC}$.
   Now, $\triangle MXB \cong \triangle MXA$ and $\triangle NXB \cong \triangle NXC$. Thus $\overline{CX} \cong \overline{AX}$, and so $\triangle CPX \cong \triangle APX$. Therefore $\overleftrightarrow{PX} \perp \overline{AC}$.

You may skip **one** of problems 2, 3, 4, 5, or 6 by crossing off the page.

3. $\boxed{\text{10 points}}$ Let $\square ABCD$ be a Saccheri quadrilateral, with right angles at A and B, and $\overline{AD} \cong \overline{BC}$ as usual. Also, let E and F be the midpoints of $\overline{AD}$ and $\overline{BC}$ respectively, and let G be the point of intersection of $\overline{EC}$ and $\overline{DF}$. Prove that $\overline{EG} \cong \overline{GF}$.



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4. $\boxed{\text{10 points}}$ Prove the Hypotenuse-Leg congruence condition. That is, suppose $\triangle ABC$ and $\triangle PQR$ are right triangles with right angles at $\angle B$ and $\angle Q$. Furthermore, suppose that $\overline{AB} \cong \overline{PQ}$ and $\overline{AC} \cong \overline{PR}$. Show that $\triangle ABC \cong \triangle PQR$. (Hint: an isosceles triangle could be helpful.)

You may skip **one** of problems 2, 3, 4, 5, or 6 by crossing off the page.

5. $\boxed{\text{10 points}}$ Prove that the Euclidean parallel postulate holds if and only if the following statement holds:

   Let lines l and m be parallel, and let line t be perpendicular to l. Then t is also perpendicular to m.

You may skip **one** of problems 2, 3, 4, 5, or 6 by crossing off the page.

6. $\boxed{\text{10 points}}$ Let D be any point interior to triangle $\triangle ABC$. Prove that the measure of $\angle BAC$ is less than the measure of $\angle BDC$.