Review Session for MAT 342: Applied Complex Analysis

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Chapter 1: Complex Numbers

- Definition, Sums, Products
  - $z = x + iy \in \mathbb{C}$, $x, y \in \mathbb{R}$, $x = \operatorname{Re} z$, $y = \operatorname{Im} z$, $i^2 = -1$
  - $z = x + iy, w = a + ib \in \mathbb{C}$
    - $z + w = (x + iy) + (a + ib) = (x + a) + i(y + b)$
    - $zw = (x + iy)(a + ib) = xa - yb + i(xb + ya)$
- Algebraic Properties
  - commutative, associative, distributive laws hold
  - $z + 0 = z$, $z \cdot 1 = z$, each $z \neq 0$ is (uniquely) invertible
  - $zw = 0 \Leftrightarrow (z = 0 \text{ or } w = 0)$
  - in other words: $\mathbb{C}$ is a field
- Vectors, Moduli, Triangle Inequality, Complex Conjugates
  - $z = x + iy \neq 0$ can be thought as being a vector from 0 to (x, y) in the plane
  - $|z| := \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}$ (euclidean length of vector from 0 to $(\operatorname{Re} z, \operatorname{Im} z)$)
  - $|z + w| \leq |z| + |w|$, $|z + w| \geq ||z| - |w||$
  - $\overline{z} = \operatorname{Re} z - i \operatorname{Im} z$, $|z|^2 = z\overline{z}$

- Exponential Form, Products, Powers
  - For $z \neq 0$: $z = r\,e^{i\theta}$, $r = |z| > 0$, $\theta \in \mathbb{R}$
  - $e^{i\theta} = \cos\theta + i\sin\theta$, thus θ is only unique up to adding a multiple of 2π.
  - $z = r\,e^{i\theta}$, $w = s\,e^{i\phi}$, then $zw = rs\,e^{i(\theta+\phi)}$
  - $z^n = r^n\,e^{in\theta}$
- Arguments, Roots of Complex Numbers
  - $\arg z = \{\theta \in \mathbb{R} \;:\; z = |z|e^{i\theta}\} = \theta_0 + 2\pi n$, $n \in \mathbb{Z}$, θ_0 one possible argument of z
  - $z = re^{i\theta} \neq 0$, all n^{th}-roots of z are $c_k = r^{\frac{1}{n}} \exp\left(i\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right)$, where $k = 0, \ldots, n-1$

- Basic Topology:
  - Neighbourhood, Deleted Neighbourhood
    - $\varepsilon > 0$, ε-neighbourhood of $z \in \mathbb{C}$ is

$$B_\varepsilon(z) = \{w \in \mathbb{C} \;:\; |w - z| < \varepsilon\}$$

    - deleted ε-neighbourhood:

$$\dot{B}_\varepsilon(z) = \{w \in \mathbb{C} \;:\; 0 < |w - z| < \varepsilon\} = B_\varepsilon(z) \setminus \{z\}$$

  - $S \subset \mathbb{C}$, $z \in \mathbb{C}$
    - z interior point of S: $\exists\,\varepsilon > 0$ s.t. $\mathcal{B}_\varepsilon(z) \subset S$
    - z exterior point of S: $\exists\,\varepsilon > 0$ s.t. $\mathcal{B}_\varepsilon(z) \cap S = \emptyset$
    - z boundary point of S: z is neither interior nor exterior point of S, i.e. $\forall\,\varepsilon > 0\,(B_\varepsilon(z) \cap S \neq \emptyset$ and $B_\varepsilon(z) \cap (\mathbb{C} \setminus S) \neq \emptyset)$, boundary of S (denoted ∂S) is the set of all boundary points of S

- Basic Topology (cont.)
  - $S \subset \mathbb{C}$
    - S is open if $\partial S \cap S = \emptyset$, i.e. each $z \in S$ is an interior point of S
    - S is closed if $\partial S \subset S$, i.e. $\mathbb{C} \setminus S$ is open
    - If S is open, S is called connected if any two points $z, w \in S$ can be joined by a polygonal line in S
    - S is a domain, if S is non-empty, open, connected
    - a domain together with some (or all) of its boundary points is a region
    - S is bounded if $S \subset B_R(0)$ for some $R > 0$

Chapter 2: Analytic Functions

- Limits

$$\lim_{z \to z_0} f(z) = w_0 \Leftrightarrow$$
$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall z \in \mathbb{C} : 0 < |z - z_0| < \delta \Rightarrow |f(z) - w_0| < \varepsilon$$

- Riemann Sphere (Limits involving ∞)
  - Riemann Sphere: Wrap $\mathbb{C}$ on a sphere sitting above the origin. Add ∞ as the north pole
  - $\lim_{z \to z_0} f(z) = \infty$, if $\lim_{z \to z_0} \frac{1}{f(z)} = 0$
  - $\lim_{z \to \infty} f(z) = w_0$, if $\lim_{z \to 0} f\left(\frac{1}{z}\right) = w_0$
  - $\lim_{z \to \infty} f(z) = \infty$, if $\lim_{z \to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0$
- Continuity, Derivatives
  - $f : D \to \mathbb{C}$ continuous at $z_0 \in D$ if $\lim_{z \to z_0} f(z) = f(z_0)$
  - $f : D \to \mathbb{C}$, $z_0 \in D$ interior point of D, f differentiable at z_0 if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists

- Cauchy-Riemann Equations (both in rectangular and polar coordinates)
  - $f(z) = u(x, y) + iv(x, y)$: $u_x = v_y$ and $u_y = -v_x$
  - $f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$: $ru_r = v_\theta$ and $u_\theta = -rv_r$
- Analytic Functions
  - f is analytic in an open set S if f is differentiable at every $z \in S$.
  - f is entire if f is analytic in $\mathbb{C}$.
  - f analytic in domain D, $f'(z) = 0$ for all $z \in D$, then f is constant
- Harmonic Functions
  - $u(x, y)$ harmonic in a domain $D \subset \mathbb{R}^2$ if

$$u_{xx}(x, y) + u_{yy}(x, y) = 0$$

for all $(x, y) \in D$
  - $f = u + iv$ analytic, then both u and v are harmonic

- Identity Theorem / Coincidence Principle
  - An analytic function in a domain D is uniquely determined by its values in a subdomain or on a line segment contained in D.
  - Most general version: $D \subset \mathbb{C}$ domain, $f, g : D \to \mathbb{C}$ analytic. If $\{z \in D \ : \ f(z) = g(z)\}$ has an accumulation point in D, then $f = g$.
- Reflection Principle: Let D be a domain which contains a segment of the real axis and whose lower half is the reflection of the upper half (i.e. $z \in D$ iff $\overline{z} \in D$). Let f be analytic in D. Then $\overline{f(z)} = f(\overline{z})$ for all $z \in D$ if and only if $f(x)$ is real for each point x on the segment.

Chapter 3: Elementary Functions

- The Exponential Function
  - $e^{x+iy} = e^x e^{iy} = e^x \cos(y) + ie^x \sin(y)$
  - $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$
  - exp is entire and $2\pi i$-periodic
- The Logarithmic Function
  - $\log z = \ln|z| + i \arg z$
  - $e^{\log z} = z$
- Branches and Derivatives of Logarithms
  - $\alpha \in \mathbb{R}$, restrict $\arg z$ so that $\alpha < \arg z < \alpha + 2\pi$, then

    $$\log z = \ln|z| + i\theta\,(|z| > 0, \alpha < \theta < \alpha + 2\pi)$$

    is a branch of the logarithm and analytic in the slit plane $\{re^{i\theta} \ : \ r > 0,\ \alpha < \theta < \alpha + 2\pi\}$ with derivative $\frac{1}{z}$
  - principle branch Log for $\alpha = -\pi$

- Power Functions
  - $z \neq 0,\ c \in \mathbb{C}$: $z^c = e^{c \log z}$
  - Given a branch of log, z^c becomes an analytic function in $\{re^{i\theta} \ : \ r > 0,\ \alpha < \theta < \alpha + 2\pi\}$ with derivative cz^{c-1}
  - principle branch of z^c: choose Log
- Trigonometric Functions, Hyperbolic Functions
  - $\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$
  - $\cos z = \frac{e^{iz} + e^{-iz}}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$
  - $\sinh z = \frac{e^{z} - e^{-z}}{2} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$
  - $\cosh z = \frac{e^{z} + e^{-z}}{2} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$
- Inverse Trigonometric and Hyperbolic Functions

Chapter 4: Integrals

- Derivatives of Functions $w : [a, b] \to \mathbb{C}$, $w(t) = u(t) + iv(t)$

$$w'(t) = u'(t) + iv'(t)$$

- Definite Integrals of such Functions

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i \int_a^b v(t)dt$$

- Contours
  - arc: $\gamma : [a, b] \to \mathbb{C}$ continuous. Also $\{\gamma(t) \ : \ t \in [a, b]\}$ is called arc
  - simple arc (Jordan arc): γ is also injective
  - simple closed curve (or Jordan curve): γ is simple except that $\gamma(a) = \gamma(b)$
  - γ is positively oriented, if it is in counterclockwise direction
  - if γ' exists on $[a, b]$ and is continuous, then *gamma* is called differentiable arc
  - if γ is differentiable and $\gamma'(t) \neq 0$ for all t, then γ is called smooth
  - A contour (or piecewise smooth arc) is an arc consisting of a finite number of smooth arcs joined end to end

- Contour Integral: $C : [a, b] \to \mathbb{C}$ contour, $f(C(t))$ piecewise continuous, then

$$\int_C f(z)dz = \int_a^b f(C(t))C'(t)dt$$

- Upper Bounds for Moduli of Contour Integrals
  - length L of contour $C : [a, b] \to \mathbb{C}$ is $L = L(C) = \int_a^b |C'(t)|dt$.
  - C contour of length L, f piecewise continuous on C with $|f(z)| \leq M$ for all $z \in C$, then

$$\left|\int_C f(z)dz\right| \leq LM$$

- Antiderivatives: f continuous in domain D. Then
  - f has antiderivative F
  - Contour integrals of f along contours lying entirely in D only depend on start and end point
  - contour integrals of f along closed contours lying entirely in D are all 0

  are equivalent

- Cauchy-Goursat Theorem: Let C be a simple closed contour and f analytic on C and inside C. Then

$$\int_C f(z)dz = 0.$$

- Simply and Multiply Connected domains
  - A domain D is simply connected if every simple closed contour lying in D only encloses points of D, i.e. "D has no holes".
  - If f is analytic in a simply connected domain D, then $\int_C f(z)dz = 0$ for every closed contour lying in D.
  - A domain D is multiply connected, if it is not simply connected.
  - C simple closed contour in counterclockwise direction, C_k, $k = 1, \ldots, n$, simple closed contours lying entirely in the interior of C, all in clockwise direction, f analytic on all of these contours and in the multiply connected domain spanned by these curves, then

$$\int_C f(z)dz + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0$$

- Cauchy Integral Formula: Let f be analytic everywhere on and inside a simple closed contour C, taken in positive sense. If z is any point interior to C, then

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta.$$

- Extended Cauchy Integral Formula: C and z as above, $n \in \mathbb{N}$, then

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta.$$

- Consequences
  - Analytic functions have derivatives of all orders.
  - Morera's theorem: Let f be continuous on a domain D. If $\int_C f(z)dz = 0$ for every closed contour C in D, then f is analytic.
  - Cauchy's inequality: f analytic inside and on a positively oriented circle C_R of radius R centred at z_0, $M_r = \max_{|z-z_0|=R} |f(z)|$, then for all $n \in \mathbb{N}$

$$\left| f^{(n)}(z_0) \right| \leq \frac{n! M_R}{R^n}.$$

- Liouville's Theorem and the Fundamental Theorem of Algebra
  - Liouville's theorem: A bounded entire function is constant.
  - Fundamental Theorem of Algebra: Every non constant complex polynomial has at least one zero.
- Maximum Modulus Principle: If f is analytic and not constant in a given domain D, then $|f(z)|$ has no maximum value in D.

Chapter 5: Series

- Sequences, Series, Convergence
  - sequence $(z_n)_{n\in\mathbb{N}}$ converges to z if

$$\forall \varepsilon > 0\, \exists n_\varepsilon \in \mathbb{N}\, \forall n \geq n_\varepsilon : |z_n - z| < \varepsilon$$

  - series: $\sum_{n=1}^{\infty} z_n$, $S_N = \sum_{n=1}^{N} z_n$. Series converges to S if $(S_N)_{N\in\mathbb{N}}$ converges to S
  - series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |z_n|$ converges
  - power series: $\sum_{n=0}^{\infty} z_n(z - z_0)^n$
- Taylor Series
  - f analytic in disk $B_{R_0}(z_0)$, then

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \quad \text{for all} \quad z \in B_{R_0}(z_0)$$

    where

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

  - if $z_0 = 0$, the Taylor Series is called Maclaurin Series

- Laurent Series: f analytic in $\{z \in \mathbb{C} \ : \ R_1 < |z - z_0| < R_2\}$, C simple closed, positively oriented contour in the annulus, then for z in that annulus

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz \quad \text{and} \quad b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz$$

- Absolute and Uniform Convergence of Power Series
  - There exists a largest ball in which a power series converges.
  - If a power series converges at $z_1 \neq z_0$, then it is absolutely convergent for every z with $|z - z_0| < |z_1 - z_0|$.
  - R radius of convergence, $R_1 < R$, then the power series is uniformly convergent on $\{z \in \mathbb{C} \ : \ |z - z_0| \leq R_1\}$.
  - Uniform convergence: The choice of n_ε in the convergence statement does not depend on the point z where convergence is investigated.

- Further Properties of Power Series $S(z) = \sum_{n_0}^{\infty} a_n (z - z_0)^n$
  - Power series represent continuous functions on their disk of convergence.
  - Power series are analytic on their disk of convergence with

$$S'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}.$$

  - The integral of a power series along some contour C inside the disk of convergence is

$$\int_C S(z) dz = \sum_{n=0}^{\infty} a_n \int_C (z - z_0)^n dz.$$

  - Power series representations are unique.

Chapter 6: Residues and Poles

- Isolated Singular Points: A singular point z_0 of an analytic function f is isolated if there exists some $\varepsilon > 0$ such that there is no other singular point in $\dot{B}_\varepsilon(z_0)$.
- Residues: z_0 isolated singular point of an analytic function f,

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

  Laurent Series of f in $\dot{B}_\varepsilon(z_0)$. The coefficient b_1 is called residue of f at z_0

$$\operatorname*{Res}_{z=z_0} f(z) = b_1 = \frac{1}{2\pi i}\int_C f(z)dz.$$

- Cauchy's Residue Theorem: Let C be a simple closed contour, described in positive sense. If f is analytic inside and on C except for a finite number of singular points z_k $(k = 1, \ldots, n)$ inside C, then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

- Residue at infinity

$$\operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

- Types of Isolated Singular Points: z_0 isolated singular point of f
  - Laurent Series $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$
  - Principle Part of Laurent Series: $\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$
  - Removable: $b_n = 0$ for all $n \in \mathbb{N}$
  - Essential: infinitely many $b_n \neq 0$
  - Pole of Order m: $b_m \neq 0$, $b_n = 0$ for all $n > m$
- Residues at Poles: z_0 pole of order m of f
  - There exists a function ϕ which is analytic at z_0 and $\phi(z_0) \neq 0$ such that

$$f(z) = \frac{\phi(z)}{(z-z_0)^m}.$$

  -

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}.$$

- Residues at Poles (cont.)
  - If $f(z) = \frac{p(z)}{q(z)}$, $p(z_0) \neq 0$, $q(z_0) = 0$, $q'(z_0) \neq 0$, then $m = 1$ and

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{p(z_0)}{q'(z_0)}.$$

- Zeros of Analytic Functions
  - z_0 is a zero of order m of f if

$$f(z_0) = f'(z_0) = \ldots = f^{(m-1)}(z_0) = 0 \quad \text{but} \quad f^{(m)}(z_0) \neq 0.$$

  - There exists a function ϕ which is analytic at z_0 and $\phi(z_0) \neq 0$ such that

$$f(z) = (z-z_0)^m \phi(z).$$

  - Zeros of analytic functions are always isolated by the coincidence principle, unless f is constantly zero.

- Zeros and Poles: Suppose that p and q are analytic at z_0, $p(z_0) \neq 0$, q has a zero of order m at z_0. Then $\frac{p(z)}{q(z)}$ has a pole of order m at z_0.
- Behaviour of Functions new Isolated Singular Points: z_0 isolated singular point of f
  - If z_0 is removable, then f is bounded and analytic in $\dot{B}_\varepsilon(z_0)$ for some $\varepsilon > 0$. Also: If a function f is analytic and bounded in $\dot{B}_\varepsilon(z_0)$, then either f is analytic at z_0 or z_0 is removable.
  - If z_0 is essential, then f assumes values arbitrarily close to any given number in any deleted neighbourhood of z_0 (Casorati-Weierstraß).
  - If z_0 is a pole of order m, then

$$\lim_{z \to z_0} f(z) = \infty.$$

Chapter 7: Applications of Residues

- Evaluation of Improper Integrals
  - If $\int_{-\infty}^{\infty} f(x)dx$ converges, then the Cauchy principle value $P.V. \int_{-\infty}^{\infty} f(x)dx$ exists.
  - The inverse is in general not true! But if f is even ($f(x) = f(-x)$), then the inverse holds.
  - Idea: Assume that $f(x) = \frac{p(x)}{q(x)}$, p and q do not share a common factor, q has no real zero but at least one zero in the upper half plane. Let $z_1, \ldots, z_n$ be the zeros of q in the upper half plane. Choose $R > 0$ so big that $|z_j| < R$ for all j. Let C_R be the semicircle of radius R in the upper half plane taken in positive sense and let C be the contour consisting of the interval $[-R, R]$ and C_R, taken in positive sense. Then

$$\int_{-R}^{R} f(x)dx = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) - \int_{C_R} f(z)dz.$$

  - If $\lim_{R\to\infty} \int_{C_R} f(z)dz = 0$ and f is even, we are done.

- Improper Integrals from Fourier Analysis
  - Want to compute integrals of the form ($a > 0$)

$$\int_{-\infty}^{\infty} f(x)\cos(ax)dx \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)\sin(ax)dx.$$

  - Caution: Same idea as on previous slide does not work. Both sin and cos are unbounded in the upper half plane!
  - Solution: $e^{iax} = \cos(ax) + i\sin(ax)$. Thus,

$$\int_{-R}^{R} f(x)\cos(ax)dx + i\int_{-R}^{R} f(x)\sin(ax)dx = \int_{-R}^{R} f(x)e^{iax}dx.$$

    Also for $z = x + iy$ in the upper half plane

$$|e^{iaz}| = |e^{iax-ay}| = e^{-ay} \leq 1.$$

    Hence, compute the last integral!

Outlook

- Argument Principle
- Rouché's Theorem
- Conformal Mappings
- Riemann Mapping Theorem