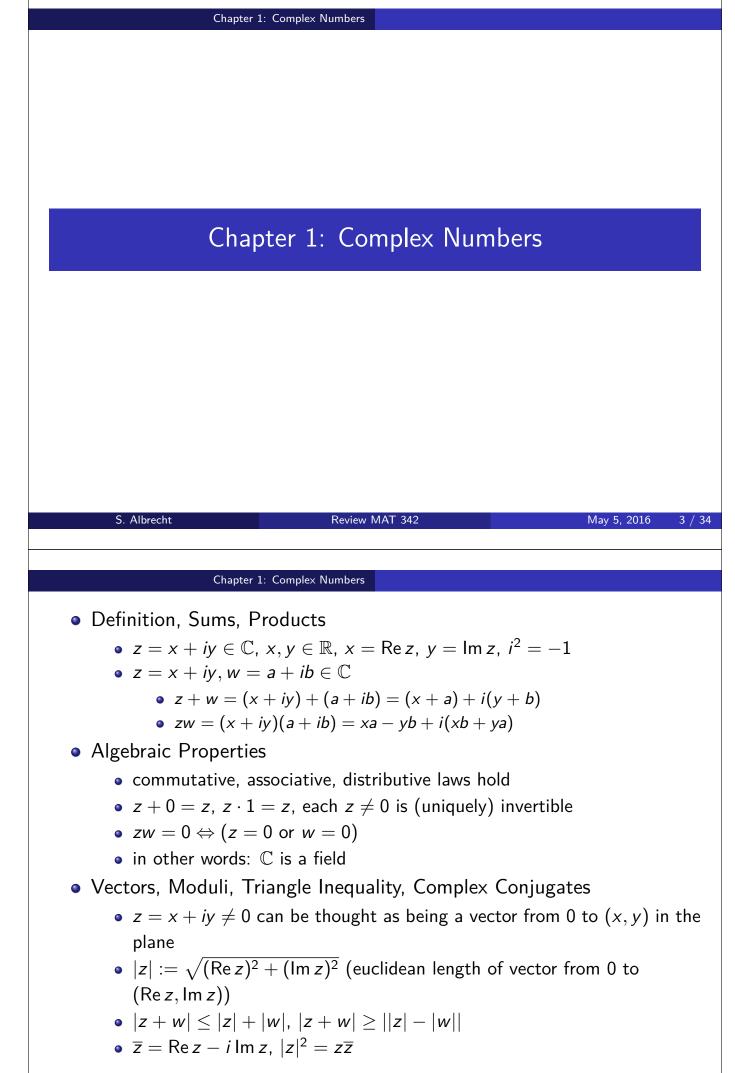
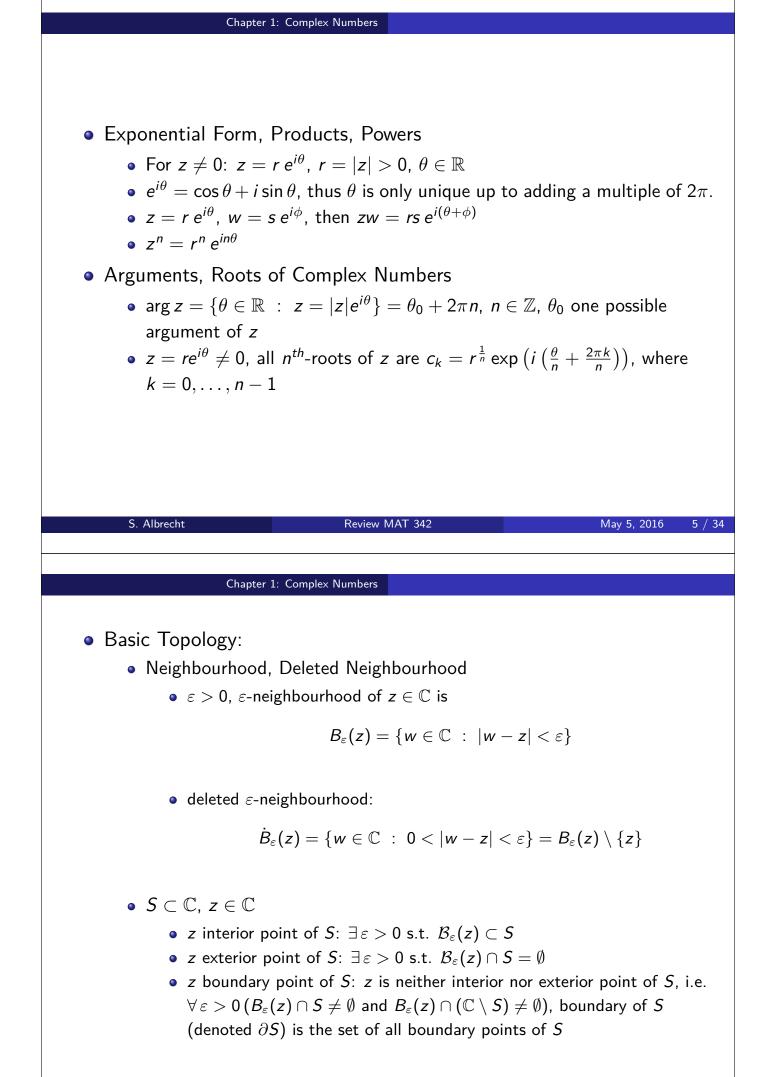
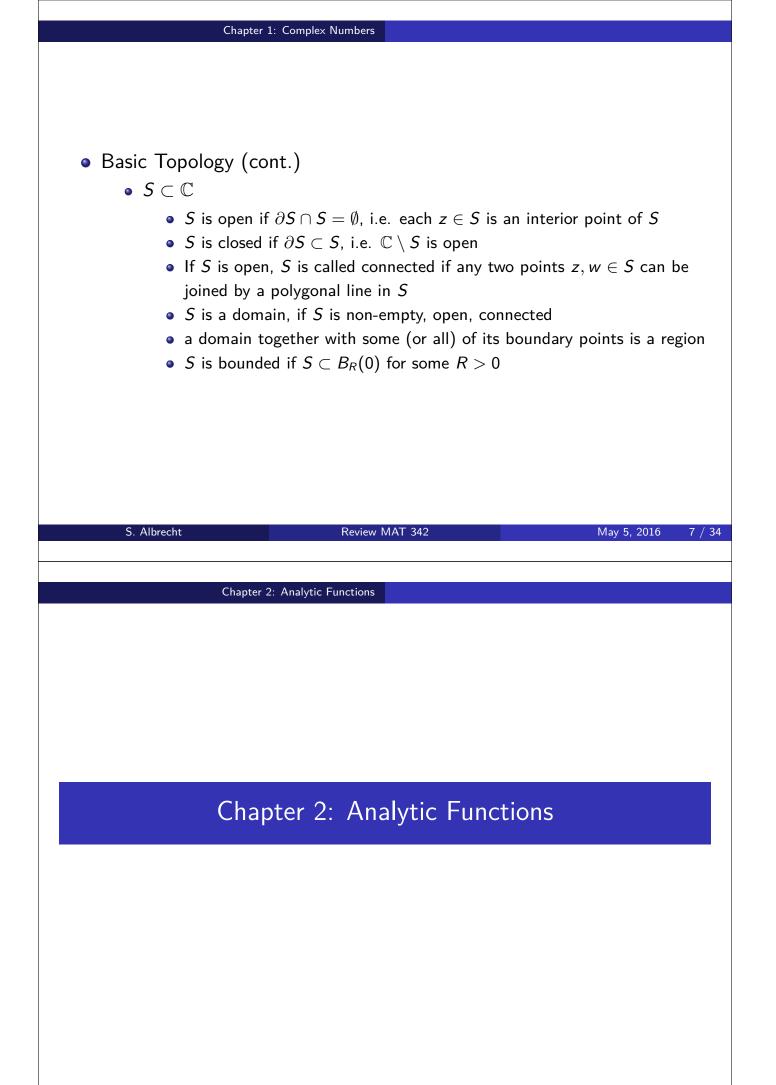
Review Session for MAT 342: Applied Complex Analysis		
Simon Albrecht		
Stony Brook, May 5, 2016		
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Limits

$$\lim_{z \to z_0} f(z) = w_0 \Leftrightarrow$$

$$\forall \varepsilon > 0 \exists \delta > 0 \forall z \in \mathbb{C} : 0 < |z - z_0| < \delta \Rightarrow |f(z) - w_0| < \varepsilon$$

• Riemann Sphere (Limits involving ∞)

- $\bullet\,$ Riemann Sphere: Wrap $\mathbb C$ on a sphere sitting above the origin. Add $\infty\,$ as the north pole
- $\lim_{z\to z_0} f(z) = \infty$, if $\lim_{z\to z_0} \frac{1}{f(z)} = 0$
- $\lim_{z\to\infty} f(z) = w_0$, if $\lim_{z\to0} f'(\frac{1}{z}) = w_0$
- $\lim_{z\to\infty} f(z) = \infty$, if $\lim_{z\to0} \frac{1}{f(\frac{1}{z})} = 0$
- Continuity, Derivatives
 - $f: D \to \mathbb{C}$ continuous at $z_0 \in D$ if $\lim_{z \to z_0} f(z) = f(z_0)$
 - $f: D \to \mathbb{C}$, $z_0 \in D$ interior point of D, f differentiable at z_0 if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists

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Chapter 2: Analytic Functions

Cauchy-Riemann Equations (both in rectangular and polar coordinates)

•
$$f(z) = u(x, y) + iv(x, y)$$
: $u_x = v_y$ and $u_y = -v_x$

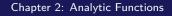
•
$$f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$$
: $ru_r = v_{\theta}$ and $u_{\theta} = -rv_r$

- Analytic Functions
 - f is analytic in an open set S if f is differentiable at every $z \in S$.
 - f is entire if f is analytic in \mathbb{C} .
 - f analytic in domain D, f'(z) = 0 for all $z \in D$, then f is constant
- Harmonic Functions
 - u(x,y) harmonic in a domain $D \subset \mathbb{R}^2$ if

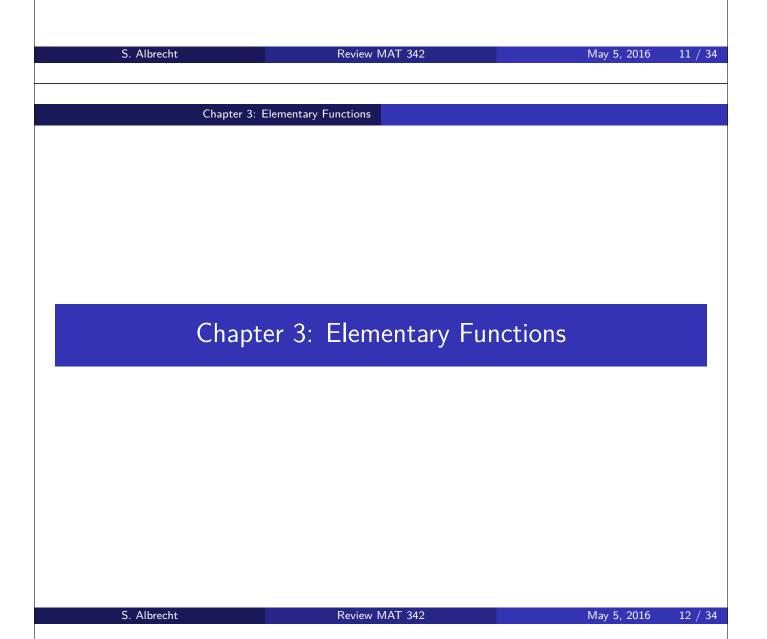
$$u_{xx}(x,y)+u_{yy}(x,y)=0$$

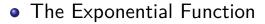
for all $(x, y) \in D$

• f = u + iv analytic, then both u and v are harmonic



- Identity Theorem / Coincidence Principle
 - An analytic function in a domain *D* is uniquely determined by its values in a subdomain or on a line segment contained in *D*.
 - Most general version: $D \subset \mathbb{C}$ domain, $f, g : D \to \mathbb{C}$ analytic. If $\{z \in D : f(z) = g(z)\}$ has an accumulation point in D, then f = g.
- Reflection Principle: Let D be a domain which contains a segment of the real axis and whose lower half is the reflection of the upper half (i.e. z ∈ D iff z̄ ∈ D). Let f be analytic in D. Then f(z) = f(z̄) for all z ∈ D if and only if f(x) is real for each point x on the segment.





- $e^{x+iy} = e^x e^{iy} = e^x \cos(y) + ie^x \sin(y)$
- $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$
- exp is entire and $2\pi i$ -periodic
- The Logarithmic Function
 - $\log z = \ln |z| + i \arg z$
 - $e^{\log z} = z$
- Branches and Derivatives of Logarithms
 - $\alpha \in \mathbb{R}$, restrict arg z so that $\alpha < \arg z < \alpha + 2\pi$, then

$$\log z = \ln |z| + i\theta (|z| > 0, \alpha < \theta < \alpha + 2\pi)$$

is a branch of the logarithm and analytic in the slit plane

 $\{re^{i heta} \; : \; r > 0, \, lpha < heta < lpha + 2\pi\}$ with derivative $rac{1}{z}$

• principle branch Log for $\alpha=-\pi$

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Chapter 3: Elementary Functions

Power Functions

- $z \neq 0$, $c \in \mathbb{C}$: $z^c = e^{c \log z}$
- Given a branch of log, z^c becomes an analytic function in { $re^{i\theta}$: r > 0, $\alpha < \theta < \alpha + 2\pi$ } with derivative cz^{c-1}
- principle branch of z^c : choose Log
- Trigonometric Functions, Hyperbolic Functions

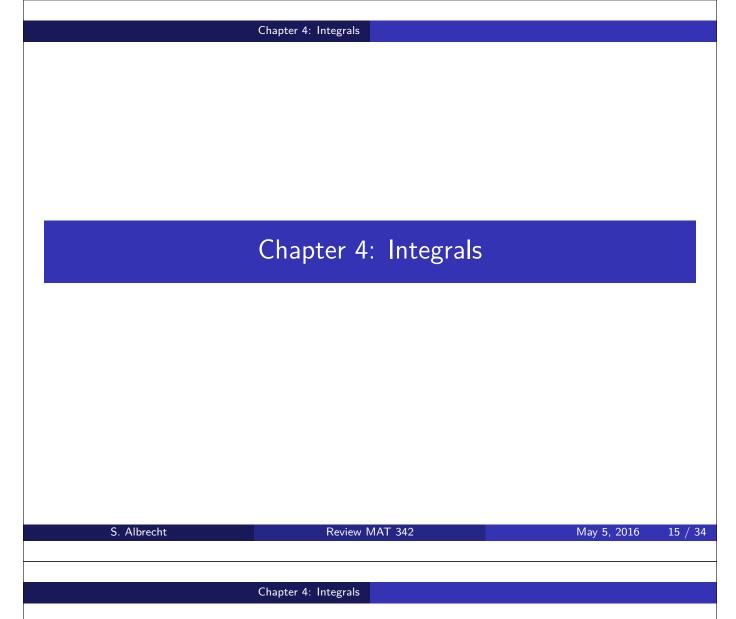
•
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

•
$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

•
$$\sinh z = \frac{e^z - e^{-z}}{2} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

•
$$\cosh z = \frac{e^z + e^{-z}}{2} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$

• Inverse Trigonometric and Hyperbolic Functions



• Derivatives of Functions $w: [a, b]
ightarrow \mathbb{C}$, w(t) = u(t) + iv(t)

$$w'(t) = u'(t) + iv'(t)$$

• Definite Integrals of such Functions

$$\int_{a}^{b} w(t)dt = \int_{a}^{b} u(t)dt + i \int_{a}^{b} v(t)dt$$

Contours

- arc: $\gamma : [a, b] \to \mathbb{C}$ continuous. Also $\{\gamma(t) : t \in [a, b]\}$ is called arc
- simple arc (Jordan arc): γ is also injective
- simple closed curve (or Jordan curve): γ is simple except that $\gamma(a) = \gamma(b)$
- γ is positively oriented, if it is in counterclockwise direction
- if γ' exists on [a,b] and is continuous, then gamma is called differentiable arc
- if γ is differentiable and $\gamma'(t) \neq 0$ for all t, then γ is called smooth
- A contour (or piecewise smooth arc) is an arc consisting of a finite number of smooth arcs joined end to end

Chapter 4: Integrals

Contour Integral: C : [a, b] → C contour, f(C(t)) piecewise continuous, then

$$\int_{C} f(z) dz = \int_{a}^{b} f(C(t))C'(t) dt$$

• Upper Bounds for Moduli of Contour Integrals

- length L of contour $C : [a, b] \to \mathbb{C}$ is $L = L(C) = \int_a^b |C'(t)| dt$.
- C contour of length L, f piecewise continuous on C with $|f(z)| \le M$ for all $z \in C$, then

$$\left|\int_{C}f(z)dz\right|\leq LM$$

- Antiderivatives: f continuous in domain D. Then
 - f has antiderivative F
 - Contour integrals of *f* along contours lying entirely in *D* only depend on start and end point
 - contour integrals of f along closed contours lying entirely in D are all 0 are equivalent

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Chapter 4: Integrals

• Cauchy-Goursat Theorem: Let C be a simple closed contour and f analytic on C and inside C. Then

$$\int_C f(z)dz=0.$$

- Simply and Multiply Connected domains
 - A domain *D* is simply connected if every simple closed contour lying in *D* only encloses points of *D*, i.e. "*D* has no holes".
 - If f is analytic in a simply connected domain D, then $\int_C f(z)dz = 0$ for every closed contour lying in D.
 - A domain *D* is multiply connected, if it is not simply connected.
 - C simple closed contour in counterclockwise direction, C_k ,

k = 1, ..., n, simple closed contours lying entirely in the interior of C, all in clockwise direction, f analytic on all of these contours and in the multiply connected domain spanned by these curves, then

$$\int_C f(z)dz + \sum_{k=1}^n \int_{C_k} f(z)dz = 0$$

 Cauchy Integral Formula: Let f be analytic everywhere on and inside a simple closed contour C, taken in positive sense. If z is any point interior to C, then

$$f(z) = rac{1}{2\pi i} \int_C rac{f(\zeta)}{\zeta - z} d\zeta.$$

• Extended Cauchy Integral Formula: C and z as above, $n \in \mathbb{N}$, then

$$f^{(n)}(z)=\frac{n!}{2\pi i}\int_C\frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta.$$

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Consequences

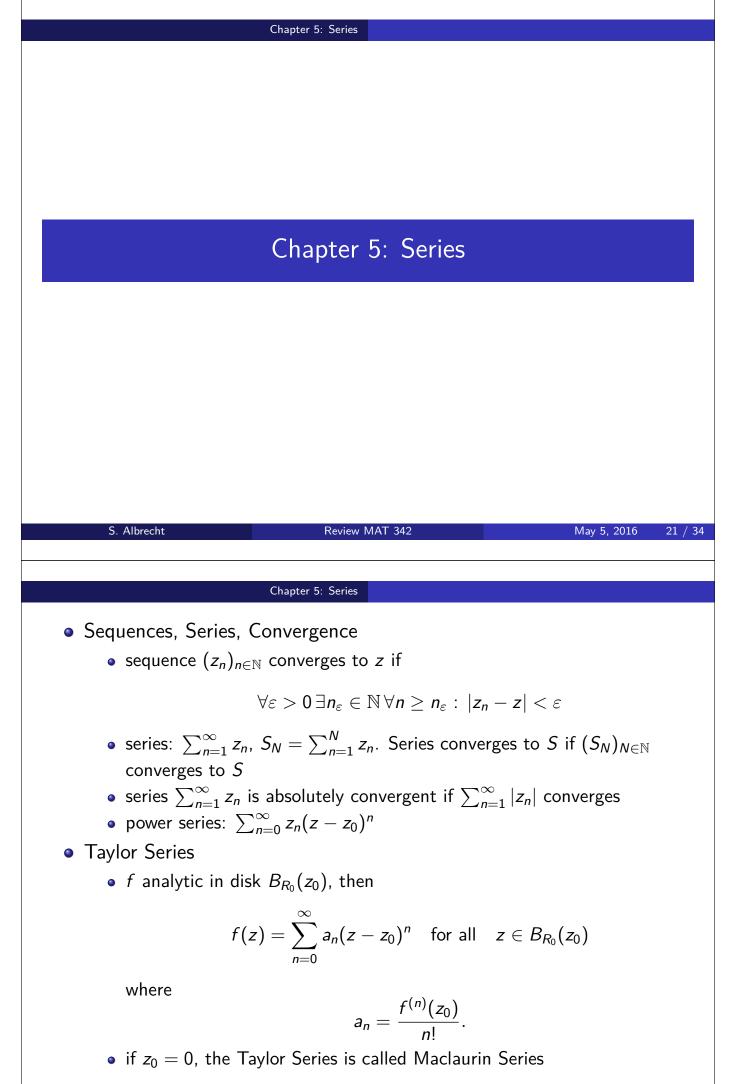
• Analytic functions have derivatives of all orders.

Chapter 4: Integrals

- Morera's theorem: Let f be continuous on a domain D. If $\int_C f(z)dz = 0$ for every closed contour C in D, then f is analytic.
- Cauchy's inequality: f analytic inside and on a positively oriented circle C_R of radius R centred at z_0 , $M_r = \max_{|z-z_0|=R} |f(z)|$, then for all $n \in \mathbb{N}$

$$\left|f^{(n)}(z_0)\right|\leq \frac{n!M_R}{R^n}.$$

- Liouville's Theorem and the Fundamental Theorem of Algebra
 - Liouville's theorem: A bounded entire function is constant.
 - Fundamental Theorem of Algebra: Every non constant complex polynomial has at least one zero.
- Maximum Modulus Principle: If f is analytic and not constant in a given domain D, then |f(z)| has no maximum value in D.



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Chapter 5: Series

• Laurent Series: f analytic in $\{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}$, C simple closed, positively oriented contour in the annulus, then for z in that annulus

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

where

$$a_n = rac{1}{2\pi i} \int_C rac{f(z)}{(z-z_0)^{n+1}} dz \quad ext{and} \quad b_n = rac{1}{2\pi i} \int_C rac{f(z)}{(z-z_0)^{-n+1}} dz$$

- Absolute and Uniform Convergence of Power Series
 - There exists a largest ball in which a power series converges.
 - If a power series converges at z₁ ≠ z₀, then it is absolutely convergent for every z with |z - z₀| < |z₁ - z₀|.
 - R radius of convergence, R₁ < R, then the power series is uniformly convergent on {z ∈ C : |z − z₀| ≤ R₁}.
 - Uniform convergence: The choice of n_{ε} in the convergence statement does not depend on the point z where convergence is investigated.

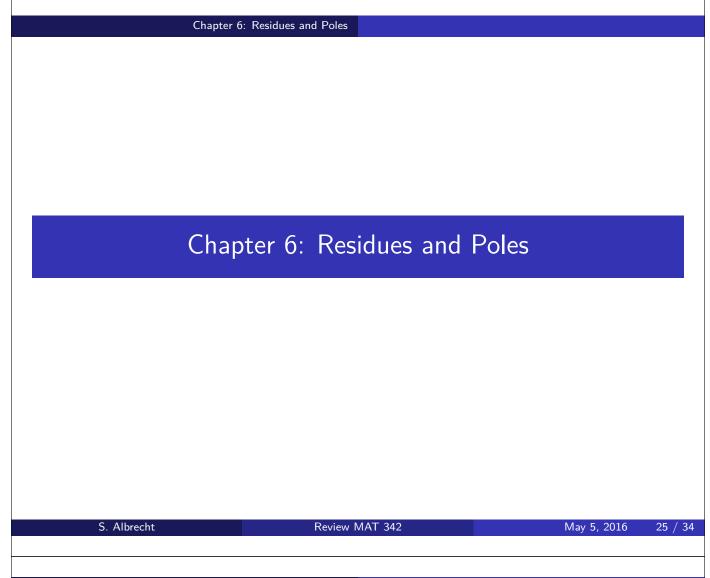
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	Chapter 5: Series		
• Further Properties	of Power Series $S(z) = \sum$	$\sum_{n_0}^{\infty} a_n (z-z_0)^n$	
	present continuous functions	ő	
convergence.			
 Power series are 	analytic on their disk of co	onvergence with	

$$S'(z)=\sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}.$$

• The integral of a power series along some contour *C* inside the disk of convergence is

$$\int_C S(z)dz = \sum_{n=0}^{\infty} a_n \int_C (z-z_0)^n dz.$$

• Power series representations are unique.



Chapter 6: Residues and Poles

- Isolated Singular Points: A singular point z₀ of an analytic function f is isolated if there exists some ε > 0 such that there is no other singular point in B_ε(z₀).
- Residues: z_0 isolated singular point of an analytic function f,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

Laurent Series of f in $\dot{B}_{\varepsilon}(z_0)$. The coefficient b_1 is called residue of f at z_0

$$\operatorname{Res}_{z=z_0} f(z) = b_1 = \frac{1}{2\pi i} \int_C f(z) dz.$$

 Cauchy's Residue Theorem: Let C be a simple closed contour, described in positive sense. If f is analytic inside and on C except for a finite number of singular points z_k (k = 1,..., n) inside C, then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^n \operatorname{Res}_{z=z_k} f(z).$$

 Residue at infinity $\operatorname{Res}_{z=\infty} f(z) = -\operatorname{Res}_{z=0} \left| \frac{1}{z^2} f\left(\frac{1}{z}\right) \right|$ • Types of Isolated Singular Points: z_0 isolated singular point of f• Laurent Series $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$ • Principle Part of Laurent Series: $\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ • Removable: $b_n = 0$ for all $n \in \mathbb{N}$ • Essential: infinitely many $b_n \neq 0$ • Pole of Order *m*: $b_m \neq 0$, $b_n = 0$ for all n > m• Residues at Poles: z_0 pole of order m of f• There exists a function ϕ which is analytic at z_0 and $\phi(z_0) \neq 0$ such that $f(z) = \frac{\phi(z)}{(z-z_0)^m}.$ $\operatorname{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}.$ Review MAT 342 S. Albrecht May 5, 2016 27 / 34 Chapter 6: Residues and Poles Residues at Poles (cont.) • If $f(z) = \frac{p(z)}{a(z)}$, $p(z_0) \neq 0$, $q(z_0) = 0$, $q'(z_0) \neq 0$, then m = 1 and $\operatorname{Res}_{z=z_0} f(z) = \frac{p(z_0)}{q'(z_0)}.$ Zeros of Analytic Functions • *z*₀ is a zero of order *m* of *f* if $f(z_0) = f'(z_0) = \ldots = f^{(m-1)}(z_0) = 0$ but $f^{(m)}(z_0) \neq 0$. • There exists a function ϕ which is analytic at z_0 and $\phi(z_0) \neq 0$ such that $f(z) = (z - z_0)^m \phi(z).$

Chapter 6: Residues and Poles

• Zeros of analytic functions are always isolated by the coincidence principle, unless *f* is constantly zero.

- Zeros and Poles: Suppose that p and q are analytic at z_0 , $p(z_0) \neq 0$, q has a zero of order m at z_0 . Then $\frac{p(z)}{q(z)}$ has a pole of order m at z_0 .
- Behaviour of Functions new Isolated Singular Points: z₀ isolated singular point of f
 - If z₀ is removable, then f is bounded and analytic in B_ε(z₀) for some ε > 0. Also: If a function f is analytic and bounded in B_ε(z₀), then either f is analytic at z₀ or z₀ is removable.
 - If z_0 is essential, then f assumes values arbitrarily close to any given number in any deleted neighbourhood of z_0 (Casorati-Weierstraß).
 - If z_0 is a pole of order m, then

$$\lim_{z\to z_0}f(z)=\infty.$$

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Chapter 7: Applications of Residues

Chapter 7: Applications of Residues

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- Evaluation of Improper Integrals
 - If $\int_{-\infty}^{\infty} f(x) dx$ converges, then the Cauchy principle value $P.V. \int_{-\infty}^{\infty} f(x) dx$ exists.
 - The inverse is in general not true! But if f is even (f(x) = f(-x)), then the inverse holds.
 - Idea: Assume that $f(x) = \frac{p(x)}{q(x)}$, p and q do not share a common factor, q has no real zero but at least one zero in the upper half plane. Let z_1, \ldots, z_n be the zeros of q in the upper half plane. Choose R > 0 so big that $|z_j| < R$ for all j. Let C_R be the semicircle of radius R in the upper half plane taken in positive sense and let C be the contour consisting of the interval [-R, R] and C_R , taken in positive sense. Then

$$\int_{-R}^{R} f(x)dx = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z=z_{k}} f(z) - \int_{C_{R}} f(z)dz$$

• If $\lim_{R\to\infty}\int_{\mathcal{C}_R}f(z)dz=0$ and f is even, we are done.

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Chapter 7: Applications of Residues

• Improper Integrals from Fourier Analysis

• Want to compute integrals of the form (a > 0)

$$\int_{-\infty}^{\infty} f(x) \cos(ax) dx \quad \text{and} \quad \int_{-\infty}^{\infty} f(x) \sin(ax) dx.$$

- Caution: Same idea as on previous slide does not work. Both sin and cos are unbounded in the upper half plane!
- Solution: $e^{iax} = \cos(ax) + i\sin(ax)$. Thus,

$$\int_{-R}^{R} f(x)\cos(ax)dx + i\int_{-R}^{R} f(x)\sin(ax)dx = \int_{-R}^{R} f(x)e^{iax}dx.$$

Also for z = x + iy in the upper half plane

$$|e^{iaz}| = |e^{iax-ay}| = e^{-ay} \le 1.$$

Hence, compute the last integral!

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