Name: ID:

MAT342 Solutions to Quiz 4

Wednesday, May 1

1. Calculate $\displaystyle\int_{\mathcal{C}} \frac{2z-3}{z^2(2z^2+9)}\,dz$ where $\mathcal{C}$ is the positively oriented circle $\{z : |z|=2\}$.
For full credit, you must fully justify your answer.

The function $\dfrac{2z-3}{z^2(2z^2+9)}$ is singular at $z=0$ and $z=\pm 3i/\sqrt{2}$; only $z=0$ lies inside the contour $\mathcal{C}$, and $f(z)$ has a pole of order 2 at $z=0$.

Let $\phi(z) = (2z-3)/(2z^2+9)$. Then we can calculate the residue at 0 as

$$\operatorname*{Res}_{z=0} f(z) = \phi'(0) = \left.\frac{2(2z^2+9)-(2z-3)(4z)}{(2z^2+9)^2}\right|_{z=0} = \frac{2}{9}\ .$$

Thus, the value of the integral is $2\pi i \operatorname*{Res}_{z=0} f(z) = \dfrac{4\pi i}{9}$.

2. Calculate $\displaystyle\int_0^\infty \frac{x\sin 2x}{x^2+3}\,dx$. For full credit, you must justify all steps in your answer.

Let $\mathcal{C}_R$ be the positively oriented semi-circle $|z|=R$ with $\operatorname{Im} z \geq 0$, let L_R be the segment of the real axis from $-R$ to R, and let $\mathcal{C}$ be the contour $\mathcal{C}_R$ followed by L_R.

When $R > \sqrt{3}$, we have

$$\int_{\mathcal{C}} \frac{z e^{2iz}}{z^2+3}\,dz = 2\pi i \operatorname*{Res}_{z=i\sqrt{3}} \frac{z e^{2iz}}{z^2+3} = 2\pi i \frac{i\sqrt{3}e^{-2\sqrt{3}}}{2i\sqrt{3}} = i\pi e^{-2\sqrt{3}}\ .$$

Note also that for any $R > \sqrt{3}$

$$\operatorname{Im}\left(\int_{\mathcal{C}} \frac{z e^{2iz}}{z^2+3}\,dz\right) = \operatorname{Im}\left(\int_{\mathcal{C}_R} \frac{z e^{2iz}}{z^2+3}\,dz + \int_{-R}^{R} \frac{x e^{2ix}}{x^2+3}\,dx\right)\ ,$$

and $\displaystyle\lim_{R\to\infty}\int_{\mathcal{C}_R} \frac{z e^{2iz}}{z^2+3}\,dz = 0$ since $\left|\dfrac{z}{z^2+3}\right| \leq \dfrac{R}{R^2-3}$ for $|z|=R$. By Jordan's lemma, this implies that the integral over $\mathcal{C}_R$ goes to 0 as $R\to\infty$.

Combining the two equations and taking the limit as $R\to\infty$, we obtain

$$\pi e^{-2\sqrt{3}} = \int_{-\infty}^{\infty} \frac{x\sin 2x}{x^2+3}\,dx\ .$$

Finally, since the integrand is an even function

$$\int_0^\infty \frac{x\sin 2x}{x^2+3}\,dx = \frac{\pi}{2\,e^{2\sqrt{3}}}\ .$$