Name: ID:

MAT342 Solutions to Quiz 3

Wednesday, April 10

1. Let $f(z) = \dfrac{(z^2-2z)\sin z - z^2 - 6z - 8}{z^3 - 4z} = \dfrac{2}{z} + \dfrac{\sin z}{z+2} - \dfrac{3}{z-2}$

and let $\mathcal{C}$ be the positively oriented circle $\{z : |z-1| = 2\}$. Calculate $\displaystyle\int_{\mathcal{C}} f(z)\,dz$.

The function $f(z)$ has singularities for $z \in \{-2, 0, 2\}$, and the contour $\mathcal{C}$ surrounds $z = 0$ and $z = 2$. Using the Cauchy-Goursat theorem, we can decompose the integral as

$$\int_{\mathcal{C}} f(z)\,dz = \int_{\mathcal{C}_0} \frac{2\,dz}{z} - \int_{\mathcal{C}_2} \frac{3\,dz}{z-2} = 4\pi i - 6\pi i = -2\pi i\ ,$$

where $\mathcal{C}_0$ is a small, positively oriented circle around 0 and $\mathcal{C}_2$ is a small, positively oriented circle around 2. Evaluating the integrals can be done by applying the Cauchy integral formula.

2. Write the Laurent series for $g(z) = z^2\cos(1/z)$. For what $z \in \mathbb{C}$ does this series converge to an analytic function?

To do this, use the Maclaurin series for $\cos(w)$ with $w = 1/z$ and multiply each term by z^2. That is,

$$z^2\cos(1/z) = z^2\sum_{n=0}^{\infty}(-1)^n\frac{(1/z)^{2n}}{(2n)!} = z^2\left(1 - \frac{1}{2z^2} + \frac{1}{4!\,z^4} - \frac{1}{6!\,z^6} + \dots\right)$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!\,z^{2n-2}} = z^2 - \frac{1}{2} + \frac{1}{4!\,z^2} - \frac{1}{6!\,z^4} + \dots$$

Since $\cos(w)$ is entire, its Maclaurin series converges for all $w \in \mathbb{C}$. Consequently, $z^2\cos(1/z)$ is analytic except for $z = 0$, and its Laurent series converges for all $z \neq 0$.