MAT342 Quiz 2 Solutions

Wednesday, February 27

1. State the Cauchy-Riemann equations, and define what it means for a function f to be analytic at a point $z_0 \in \mathbb{C}$.

If $f(x+iy) = u(x,y) + iv(x,y)$, then the Cauchy-Riemann equations are merely that the partials of u and v exist and satisfy

$$u_x = v_y \qquad \text{and} \qquad u_y = v_x \ .$$

A function f is analytic at z_0 when $f'(z)$ exists for all z in some neighborhood of z_0.

If the derivative of f exists at z, the Cauchy-Riemann equations must hold at z. However, just having the Cauchy-Riemann equations satisfied at z_0 is not enough to guarantee analyticity. If the partials of u and v are continuous at z_0 and the Cauchy-Riemann equations hold, that is sufficient to know that f is analytic at z_0.

2. Consider the function given by $\quad f(z) = \log\left(e^{iz} + e^{-iz} - 2\right)$.
State a domain on which f is analytic (if one exists), and another on which f is not analytic (if one exists). Fully justify both answers.

Observe that since $\cos(z) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$, we have

$$f(z) = \log(2\cos(z) - 2) \ .$$

The logarithm is singular at 0, so to ensure that f is analytic, we must avoid those z such that $\cos(z) = 1$. This happens exactly at all integer multiples of 2π.

The question asks for *any* domain where f is analytic, and *any other* domain where f is not analytic. So a correct answer could be, for example,

- f is analytic on the open disk $|z - i| < \frac{1}{2}$
- f is not analytic on the open disk $|z| < 1$

since the first avoids all integer multiples of 2π, while the second includes 0.

However, we have to be a little bit careful about which branch of the logarithm we take. The choice above works for the principal branch of the logarithm (ie, cut along the negative real axis). You could also take, for example, the domain $\mathrm{Im}(z) > 0$, which also works for the principal branch of the logarithm.

If you want to include real values of z in your domain, you need to choose a different branch of the logarithm, since for $z \in \mathbb{R}$ we have $\cos(z) - 1 \leq 0$. For example, you want f to be analytic on a domain like $|z - 1| < 1$, you would have to choose a branch of the logarithm which includes the negative reals, for example, defined for $z \neq 0$ and $0 < \arg z < 2\pi$. There are plenty of other possibilities. . .