MAT342 Quiz 1
Solutions

1. Find all solutions to $z^4 = -16$. Fully justify your answer.

Write $z = re^{i\theta}$. We want to find all r and θ satisfying $\left(re^{i\theta}\right)^4 = -16$.
Observe that $-1 = e^{i\pi}$ and that $16 = 2^4$. Hence $r = 2$ and $4\theta = \pi + 2\pi k$ for some $k \in \mathbb{Z}$.

There are 4 solutions for θ, namely $\theta = \pi/4$, $\theta = 3\pi/4$, $\theta = 5\pi/4$ and $\theta = 7\pi/4$ (you could use $\theta = -\pi/4$ and $\theta = -3\pi/4$ if you prefer).

This means the four solutions of $z^4 = -16$ are

$$2e^{3i\pi/4} = -\sqrt{2} + i\sqrt{2}, \quad 2e^{i\pi/4} = \sqrt{2} + i\sqrt{2},$$
$$2e^{5i\pi/4} = -\sqrt{2} - i\sqrt{2}, \quad 2e^{7i\pi/4} = \sqrt{2} - i\sqrt{2}.$$

Here's another way. Since $z^4 = -16$, we know $z^2 = \pm\sqrt{-16} = \pm 4i$. So take the square root of each of *those*. Of course, we have to figure out that $\sqrt{i} = \pm\frac{1+i}{\sqrt{2}}$ and that $\sqrt{-i} = \pm i \cdot \frac{1+i}{\sqrt{2}} = \pm\frac{-1+i}{\sqrt{2}}$ and then multiply each answer by $\sqrt{4} = 2$. Those four results simplify to the four solutions above.

You might try other things. For example, you could (but *why?*) write $z = x + iy$, so $z^4 = (x+iy)^4 = x^4 - 6x^2y^2 + y^4 - 4(x^3y + xy^3)i$. Now solve the simultaneous equations $x^4 - 6x^2y^2 + y^4 = -16$, $x^3y + xy^3 = 0$. But this is a bad idea, since it leads you back to apparently needing to solve $y^4 = -16$, and you just get sad. Don't do this. (If you insist, after a ton of horrible calculation, you can discover that $x = \pm\sqrt{2}, y = \pm\sqrt{2}$ works, which yeilds the answer above.)

Or you could use de Moivre. Even this is too much work for me, so I won't. Guess I'm just lazy.

2. Consider the function $J(z) = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$, and let S be the unit circle $\{\, z \in \mathbb{C} : |\,z| = 1\,\}$. Describe the image of S under J and justify your answer. (That is, describe the set $\{\, w \in \mathbb{C} : w = J(z) \text{ for } z \in S\,\}$. Writing z in exponential notation might help.

Any point $z \in S$ can be written as $e^{i\theta}$ for some θ. In this case, $1/z = 1/e^{i\theta} = e^{-i\theta} = \bar{z}$. Hence for $z \in S$,

$$J(z) = \tfrac{1}{2}(z + \bar{z}) = \tfrac{1}{2}(2\mathrm{Re}\,(z)) = \mathrm{Re}\,(z).$$

This means the image of the unit circle under J is just the projection of the unit circle onto the real axis, that is, the interval $[-1, 1] \subset \mathbb{R} \subset \mathbb{C}$, or if you prefer,
the set $\{\, x + iy : y = 0, |\,x| \le 1\,\}$.

A lot of people figured out that $J(re^{i\theta}) = \cos\theta$ and then drew the graph of $\cos\theta$ as a function of θ (that is, a wiggly curve). But this is not the image in $\mathbb{C}$, it is the graph of a function from $\mathbb{R}$ to $\mathbb{R}$. Rather, the image is all the possible values of $\cos\theta$ as $0 \le \theta \le 2\pi$ (or $\theta \in \mathbb{R}$), that is, the real numbers between -1 and 1 inclusive. This is a line segment, not a wiggly curve.