MAT 342 Solutions to Midterm

10 pts 1. Simplify each of the following, writing your answer in the form $a + ib$ where a and b are real numbers (or sets of real numbers). Justify your answers fully.

(a) $\dfrac{1-i}{1+\sqrt{3}i}$

Solution:

$$\frac{1-i}{1+\sqrt{3}i} = \frac{(1-i)(1-\sqrt{3}i)}{(1+\sqrt{3}i)(1-\sqrt{3}i)} = \boxed{\frac{1-\sqrt{3}}{4} - i\,\frac{1+\sqrt{3}}{4}}$$

(b) $\left(\dfrac{1-i}{\sqrt{2}}\right)^i$

Solution:

$$\left(\frac{1-i}{\sqrt{2}}\right)^i = e^{i\log([1-i]/\sqrt{2})} = \exp\left(i(\ln 1 - i\tfrac{\pi}{4} + 2k\pi\, i)\right) = e^{\pi/4}e^{-2k\pi}, \qquad k \in \mathbb{Z};$$

that is, $\boxed{\{\ldots, e^{-23\pi/4}, e^{-15\pi/4}, e^{-7\pi/4}, e^{\pi/4}, e^{9\pi/4}, e^{17\pi/4}, e^{25\pi/4}, \ldots\}}$.

10 pts 2. For each of the functions below, state all values of z for which the function is analytic. (If there are none, say so). Justify your answer fully.

(a) $f(z) = \dfrac{\overline{z}}{|z|^2}$

Solution: Since $|z|^2 = z \cdot \overline{z}$, we have $f(z) = \dfrac{1}{z}$ which is $\boxed{\text{analytic on } \mathbb{C} \setminus \{0\}}$.

Of course, you can also do this by writing $f(x+iy) = \dfrac{x-iy}{x^2+y^2} = u(x,y) + iv(x,y)$, and then checking that the partials are continuous for $z \neq 0$ and the Cauchy-Riemann equations are satisfied. That is actual work, but let's do it anyway.

$$u_x = \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \qquad u_y = \frac{-2xy}{(x^2+y^2)^2}$$

$$v_x = \frac{2xy}{(x^2+y^2)^2} \qquad v_y = -\frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

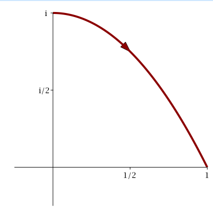
Since we have $u_x = v_y$ and $u_y = -v_x$ and the partials are continuous for $z \neq 0$, the function is $\boxed{\text{analytic for } z \neq 0}$.

(b) $g(z) = \overline{z}^2$

Solution: Write $\overline{z}^2 = (x - iy)^2 = (x^2 - y^2) - 2xy\,i = u + i\,v$. Then we have

$$u_x = 2x, \quad u_y = -2x \qquad v_x = -2y = u_y$$

and the Cauchy-Riemann equations can only be satisfied when $x = -x$ and $y = -y$; hence, the only solution is $z = 0$. Thus $g'(0) = 0$ and $g'(z)$ does not exist for $z \neq 0$. For g to be analytic, it must exist in an entire neighborhood, so $\boxed{g \text{ is nowhere analytic}}$.



10 pts

3. Let $\gamma(t) = t + (1 - t^2)i$, with $0 \leq t \leq 1$. Calculate the integral

$$\int_\gamma z^2 + \frac{1}{z}\,dz\,.$$

Solution: Observe that in the upper half-plane, the function $f(z)$ has an analytic anti-derivative $z^3/3 + \log z$ (you can choose any branch of the logarithm $\alpha < \arg z < 2\pi + \alpha$ as long as $e^{i\alpha}$ is not in the first quadrant). In particular, the principal branch works. So:

$$\int_\gamma z^2 + \frac{1}{z}\,dz = \frac{z^3}{3} + \log z\Big|_i^1 = \left(\frac{1}{3} + \operatorname{Log} 1\right) - \left(\frac{i^3}{3} + \operatorname{Log} i\right) = \boxed{\frac{1}{3} + \frac{i}{3} - \frac{\pi\,i}{2}} = \boxed{\frac{2 + (2 - 3\pi)i}{6}}\,.$$

If you insist on evaluating along the contour directly, that's OK too. Have fun.

Well, OK, I'll do it here but I don't want to.

$$\begin{aligned}
\int_\gamma z^2 + \frac{1}{z}\,dz &= \int_0^1 f(\gamma(t))\gamma'(t)\,dt \\
&= \int_0^1 \left(\big(t + i(1 - t^2)\big)^2 - \frac{1}{t + i(1 - t^2)}\right)(1 - 2it)\,dt \\
&\qquad \text{some ugly algebraic calculation omitted here} \\
&= \int_0^1 2it^5 - 5t^4 - 8it^3 + 7t^2 + 4it - 1 + \frac{1 - 2it}{-it^2 + t + i} \\
&= \frac{it^6}{3} - t^5 - 2it^4 + \tfrac{7}{3}t^3 + 2it^2 - t + \operatorname{Log}(-it^2 + t + i)\Big|_0^1 \\
&= \left(\frac{i}{3} - 1 - 2i + \frac{7}{3} + 2i - 1 + \operatorname{Log} 1\right) - \operatorname{Log} i \\
&= \left(\frac{1}{3} + \frac{i}{3}\right) - \frac{\pi i}{2} = \boxed{\frac{1}{3} + \frac{i}{3} - \frac{\pi\,i}{2}}\,.
\end{aligned}$$

That was significantly less fun, but not as bad as I expected. I only had to correct my work like 6 times. Not my recommended method.

10 pts 4. Consider the domain $\mathcal{D}$ consisting of the complex plane with the positive imaginary axis removed (0 is also not included in $\mathcal{D}$).

Let f be the multivalued function $z \mapsto z^{1/4}$. *Explicitly* describe the branch of f for which $f(1) = -i$, and calculate $f(-1)$.

Solution: The function $z \mapsto z^{1/4}$ is a 4-valued function: there are four solutions to $w^4 = 1$, namely $w \in \{1, i, -1, -i\}$. Recall that branches of $z^{1/4}$ can all be written as $re^{i\theta} \mapsto \sqrt[4]{r}\, e^{i\theta/4}$, with $r > 0$ and $\alpha < \theta < \alpha + 2\pi$ for an appropriate choice of α.

Since we need a branch f_4 that takes 1 to $-i = e^{-i\pi/2}$, we must ensure that $1 = e^{-2\pi i}$. The domain $\mathcal{D}$ for f is slit along the positive imaginary axis (z with $\arg z = \pi/2 + 2k\pi$), and so we choose $k = -1$. That is, we want $\alpha = -2\pi - \pi/2 = -5\pi/2$. Hence the desired branch is

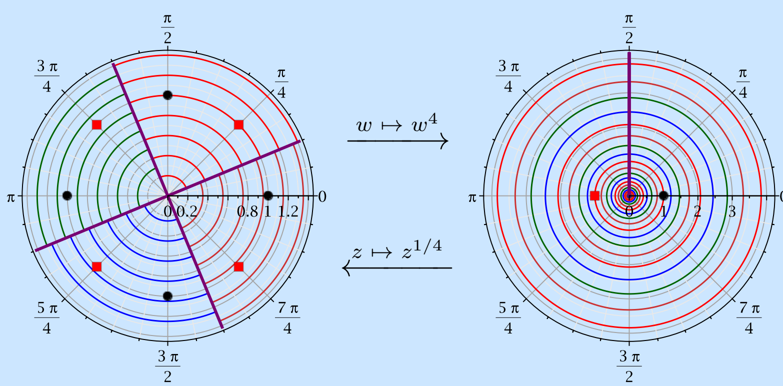
$$\boxed{f_4(re^{i\theta}) = \sqrt[4]{r}\, e^{i\theta/4} \qquad \text{with } r > 0 \text{ and } -7\pi/2 < \theta < -3\pi/2}.$$

Then for this branch we have

$$f_4(-1) = f_4(e^{-3\pi i}) = e^{-3\pi i/4} = \boxed{\frac{-1-i}{\sqrt{2}}}.$$

Various other choices of k (for example, $k = 3$, $k = 7$) will give equivalent answers; others ($k = 0$, $k = 1$, $k = 2$, $k = 5$, etc.) will not.

Let's look at this again. Shown below on the left are a series of quarter-circles (red, green, blue, and orange), and four purple radial lines. The fourth roots of 1 are marked with black disks, and the fourth roots of -1 are marked with red boxes. On the right is the image of the left-hand picture under $w \mapsto w^4$.



To get $z \mapsto z^{1/4}$, we need to go from right to left. But we need to choose a branch so the function is well-defined. The blue one is the one specified in the problem. Taking $k = 0$ gives the orange branch, $k = 1$ gives the red, and $k = 2$ gives the green. Then the pattern repeats. The most general solution could be written as

$$\boxed{f_4(re^{i\theta}) = \sqrt[4]{r}\, e^{i\theta/4} \qquad \text{with } r > 0 \text{ and } 2(4n-1)\pi - \pi/2 < \theta < 2(4n-1)\pi + 3\pi/2, \quad n \in \mathbb{Z}}.$$

10 pts 5. Let $E = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \text{ and } \operatorname{Re} z \geq 0 \}$ and describe the set $f(E)$ where $f(z) = -z^3$. As part of your answer, include a sketch of $f(E)$, using solid lines to indicate boundaries which are included and dashed lines to indicate boundaries not included. Shade the interior of $f(E)$ (if there is one).

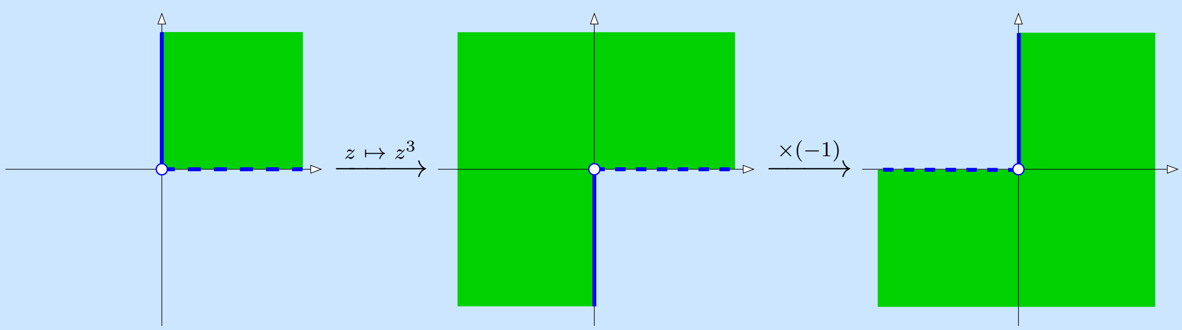
Solution: First, observe that $f(0) = 0$, which although $0 \notin E$, it will be helpful to keep track of since 0 is on the boundary of E. If we think of E in polar form, things become a bit more apparent.

For $z \neq 0$, we can write $z = re^{i\theta}$, and then we have

$$f(re^{i\theta}) = -re^{3\theta i} = e^{i\pi} \cdot re^{3\theta i} = re^{(3\theta+\pi)i} .$$

The set E can be described as the points $z \in \mathbb{C}$ with $0 < \arg z \leq \pi/2$. Then the image $f(E)$ will consist of $\boxed{\text{those } z \text{ with } \operatorname{Im} z < 0 \text{ or } \operatorname{Re} z \geq 0 \text{ (but } z \neq 0)}$. Alternatively, you can describe it as $\boxed{z \neq 0 \text{ and } -\pi < \arg z \leq \pi/2}$.

Thinking of f as the composition of $z \mapsto z^3$ followed by multiplication by -1, we have the picture below (E and its images are shown in green, with a blue boundary).



That is, first we triple the argument and cube the modulus ($z \mapsto z^3$), and then multiplying by -1 rotates the plane by a half-turn (which is the same as adding π to the argument, if you prefer to think of it that way.)

Note that while you *can* write $f(z)$ as $f(x+iy) = -x^3 + 3xy^2 + i(-3x^2y + y^3)$, it isn't terribly helpful to do so (at least not for me, nor was it for most people who tried to do the problem this way.)