Stony Brook University
Mathematics Department
Oleg Viro

Applied Complex Analysis
MAT 342
April 15, 2008

Final Exam

Examination time: 8:00-10:30 am. No electronic devices, books or notes. Show all your work.

Name SOLUTIONS

Student ID # ______

Problem #	Points/total
1	/5
2	/5
3	/10
4	/10
5	/20
6	/15
7	/15
8	/20
Total	/100

Name ________________________________

Problem 1 (5pt). Find all complex values of $1^{\sqrt{2}}$. Give the answer in the form $a + ib$.

$= 1 \in \mathbb{R}.$

OTHER COMPLEX VALUES.

Name SOLUTIONS

Problem 2 (5pt). Let $f(x+iy) = u(x,y) + iv(x,y)$ be an analytic function. Prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

SINCE f IS ANALYTIC, $\mathrm{Re}\, f$ IS HARMONIC, SO THATS DONE.

MAYBE THAT'S TOO EASY, SO BY CAUCHY RIEMANN EQ'NS,

$$u_x = v_y \quad \text{AND} \quad v_x = -u_y$$

THESE PARTIALS ARE CONTINUOUSLY DIFFERENTIABLE, SO

$$u_{xx} = v_{yx} = v_{xy} = -u_{yy}$$

SINCE MIXED PARTIALS ARE EQUAL FOR CONT. DIFF. FUNCTIONS.

THAT IS,

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 v}{\partial y^2}$$

Name SOLUTIONS

Problem 3 (10pt). Find the image of the set $S = \{(x, y) \in \mathbb{C} \,|\, 0 < x < 1\}$ under the transformation

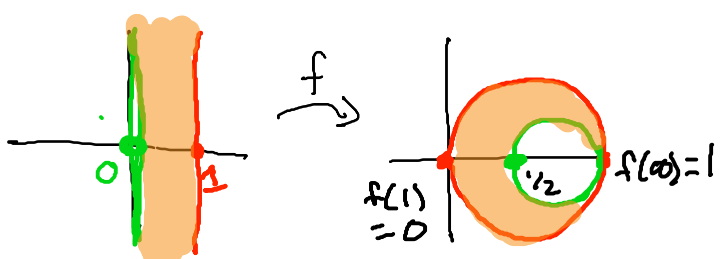
$$f(z) = \frac{z-1}{z-2}.$$

SINCE $f: \mathbb{R} \to \mathbb{R}$, $f(\bar{z}) = \frac{\bar{z}-1}{\bar{z}-2} = \frac{\overline{z-1}}{\overline{z-2}} = \overline{\left(\frac{z-1}{z-2}\right)} = \overline{f(z)}$.

SINCE IF $z \in S$, $\bar{z} \in S$, $f(S)$ MUST BE SYMMETRIC WITH RESPECT TO $\mathbb{R}$.

SINCE f IS MÖBIUS, THE IMAGE OF THE TWO LINES BOUNDING S WILL BE CIRCLES SYMMETRIC TO $\mathbb{R}$.

SINCE $f(0) = \frac{1}{2}$ AND $f(\infty) = \lim_{z \to \infty} \frac{z-1}{z-2} = 1$, THE IMAGE OF THE IMAGINARY AXIS IS THE CIRCLE WITH DIAMETER $[1/2, 1]$. SIMILARLY, SINCE $f(1) = 0$, THE IMAGE OF $\mathrm{Re}(z) = 1$ IS THE CIRCLE WITH DIAMETER $[0, 1]$. THE IMAGE OF S IS THE CRESCENT BETWEEN THESE CIRCLES.



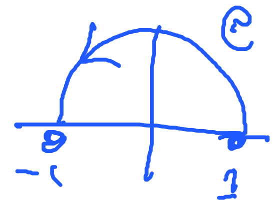
IF YOU FORGOT ABOUT MÖBIUS TRANSFORMATIONS, ONE WAY TO DO THIS IS TO WRITE $w = z - 2$,

SO $\frac{z-1}{z-2} = \frac{w+1}{w} = 1 + \frac{1}{w}$. THAT IS, f INVERTS S INTO THE UNIT CIRCLE IN w (IE, CENTERED AT $z = 2$), THEN SHIFTS BY 1 TO THE LEFT.

SAME - SAME.

YOU COULD PARAMETERIZE THE LINES BOUNDING S & COMPUTE THE IMAGE, BUT THATS REALLY ICKY & NOT FOR ME.

Name ______________________________



Problem 4 (10pt). Evaluate the integral

$$\int_C |z|\,\bar{z}\,dz,$$

where C is the boundary of the set $\{z \in \mathbb{C} \,|\, |z| \le 1,\ \operatorname{Im} z \ge 0\}$ taken in the counterclockwise direction.

WRITE $z = e^{it}$ with $0 < t < \pi$

$dz = ie^{it}dt, \quad |z| = 1, \quad \bar{z} = e^{-it}.$

$$\int_C |z|\bar{z}\,dz = \int_0^\pi 1 \cdot e^{-it} \cdot ie^{it}\,dt$$

$$= \int_0^\pi i\,dt = \pi i$$

OR YOU COULD NOTICE THAT ON C,

$|z| = 1, \quad \bar{z} = \frac{1}{z}$, SO

$$\int_C |z|\bar{z}\,dz = \int_C \frac{1}{z}\,dz = \log(-1) - \log 1$$

$$= \pi i - 0$$

$$= \pi i.$$

Name SOLUTIONS

Problem 5 (20pt). In which domains is the function $f(z) = \dfrac{1}{(z-1)(z-i)}$ represented by Laurent series in powers of z? Find the Laurent series in the unbounded domain.

F HAS SINGULARITIES AT $z=1$ AND $z=i$

SO F HAS A LAURENT SERIES ABOUT $z=0$

FOR $|z|<1$ AND FOR $|z|>1$

(WE ALSO HAVE SERIES ABOUT OTHER POINTS,
EG FOR $0<|z-1|<\sqrt{2}$ AND $|z-1|>\sqrt{2}$,
BUT LETS NOT GO THERE)

USING PARTIAL FRACTIONS, WE GET

$$\frac{1}{(z-1)(z-i)} = \frac{1+i}{2}\left(\frac{1}{z-1} - \frac{1}{z-i}\right)$$

SINCE $|z|>1$, WE USE $\dfrac{1}{1-1/z} = \sum_{n=0}^{\infty}\left(\dfrac{1}{z}\right)^n$

SO $$\frac{1}{z-1} = \frac{1}{z}\left(\frac{1}{1-1/z}\right) = \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} = \sum_{m=1}^{\infty}\frac{1}{z^m} = \frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots$$

AND $$\frac{1}{z-i} = \frac{1}{z}\left(\frac{1}{1-i/z}\right) = \frac{1}{z}\sum_{n=0}^{\infty}\frac{i^n}{z^n} = \sum_{m=1}^{\infty}\frac{i^{m-1}}{z^m} = \frac{1}{z}+\frac{i}{z^2}-\frac{1}{z^3}-\frac{i}{z^4}+\cdots$$

SO FOR $|z|>1$, WE HAVE

$$\frac{1}{(z-1)(z-i)} = \frac{1+i}{2}\sum_{m=1}^{\infty}\frac{1-i^{m-1}}{z^m}$$

$$= \left(\frac{1+i}{2}\right)\left(\frac{1-i}{z^2} + \frac{2}{z^3} + \frac{1+i}{z^4} + \frac{1-i}{z^6} + \frac{2}{z^7} + \frac{1+i}{z^8} + \cdots\right)$$

Name ________________________________

Problem 6 (15pt). Evaluate the integral

$$\oint_C \frac{z+1}{e^z+1}\,dz, \quad = \int_C f(z)\,dz$$

where C is the circle $|z| = 4$ taken in the counterclockwise direction.

$1 + e^z = 0$ EXACTLY WHEN $z = \pi i$,

WHICH LIES INSIDE C.

BY THE CAUCHY-GOURSAT THEOREM,

$$\int_C f(z)\,dz = 2\pi i \left(\operatorname*{Res}_{z=\pi i} f(z) \right)$$

$$= 2\pi i \left(\frac{\pi i + 1}{e^{\pi i}} \right) = 2\pi i(-1-\pi i)$$

$$= 2\pi^2 - 2\pi i$$

Name SOLUTIONS

Problem 7 (15pt). Find and classify all singularities of the function $f(z) = \dfrac{z-\pi}{\sin 2z}$. Find the principal part of Laurent expansion of f about $z = 2\pi$.

SINCE $\sin 2z = 0$ EXACTLY WHEN $z = \frac{k\pi}{2}$ $(k \in \mathbb{Z})$,
$f(z)$ HAS A POLE AT EVERY MULTIPLE OF $\pi/2$ EXCEPT FOR $z = \pi$ (WHERE THERE IS A REMOVABLE SINGULARITY)
SINCE $(\sin 2z)' = 2\cos 2z$ IS NONZERO AT THESE POINTS, THESE POLES ARE ALL SIMPLE.

THE PRINCIPAL PART OF THE LAURENT EXPANSION IS JUST THE NEGATIVE TERMS IN THE SERIES.
SINCE THERE IS A SIMPLE POLE AT $z = 2\pi$, THERE IS JUST ONE TERM,
WHICH IS

$$\frac{\operatorname{Res}_{z=2\pi} f}{z-\pi}$$

$$\operatorname{Res} f = \frac{2\pi - \pi}{2\cos(4\pi)} = \frac{\pi}{2},$$

SO THE DESIRED TERM IS

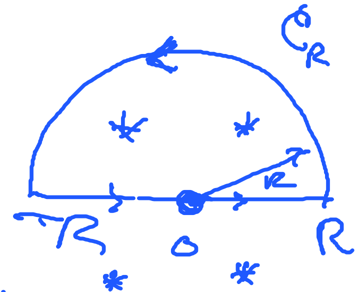
$$\frac{\pi}{2}(z-\pi)^{-1}$$

Name ______________________________

Problem 8 (20pt). Evaluate the integral

$$\int_0^\infty f(x)\,dx = \int_0^\infty \frac{x^2}{x^4+1}\,dx.$$

Explain carefully each step in your calculation.



LET C_R BE THE SEMICIRCLE $|z|=R$, $\operatorname{Im} z>0$ ORIENT COUNTERCLOCKWISE, AN L_R BE THE SEGMENT OF THE REAL AXIS FROM $-R$ TO R, AND $C = C_R + L_R$.

THEN $$\int_C \frac{z^2\,dz}{z^4+1} = \int_{C_R} \frac{z^2\,dz}{z^4+1} + \int_{-R}^{R} \frac{x^2}{x^4+1}\,dx$$

SINCE $f(z)$ IS SINGULAR INSIDE C AT $z=e^{i\pi/4}$ AND $z=e^{i3\pi/4}$

$$\int_C f(z)\,dz = 2\pi i\left(\operatorname*{Res}_{z=e^{i\pi/4}} f(z) + \operatorname*{Res}_{z=e^{i3\pi/4}} f(z)\right)$$

$$= 2\pi i\left(\frac{i}{2\sqrt{2}(i-1)} - \frac{i}{2\sqrt{2}(i+1)}\right) = \frac{\pi}{\sqrt{2}}$$

NOTE $\left|\int_{C_R} f(z)\,dz\right| < \frac{R^2}{1-R^4}\cdot \pi R$ WHICH $\to 0$ AS $R\to\infty$

SINCE f IS EVEN, $\lim\limits_{R\to\infty} \int_{-R}^{R} \frac{x^2}{x^4+1}\,dx = \int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx$

SO $$\frac{\pi}{\sqrt{2}} = \lim_{R\to\infty} \int_C f(z)\,dz = 0 + \int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx$$

AND $$\int_0^\infty \frac{x^2}{x^4+1}\,dx = \frac{\pi}{2\sqrt{2}}$$