MAT 342 Practice Final

Applied Complex Analysis

May 2019

Name: ____

Student ID: ____

Instructions:

There are 10 problems on 12 pages (plus this cover sheet) in this exam. Make sure that you have them all.

Answer all questions below. You may use the backs of pages, but in this case indicate where your work for each problem is located. You must show all of your work and provide **complete justifications** for all of your claims. Insufficient justifications will not receive full credit. Use **non-erasable pen**; do not use pencil.

Leave all answers in exact form (that is, do not approximate π, square roots, and so on.)

You may **not** use books, notes, calculators, cell phones or other electronic devices.

Problem	1	2	3	4	5	6	7	8	9	10	Total
Points	10	10	10	10	10	10	10	10	10	15	105
Score											

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[10 pts] **Problem 1.**

(i) Show that the function $\ln|z|$ is harmonic everywhere except at 0.

(ii) Consider a function $f(z) = u(z)+iv(z)$ defined in a domain D. If u and v are harmonic in D, is it true that f is analytic in D?

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[10 pts] **Problem 2.**

(i) When is an isolated singular point of an analytic function called removable?

(ii) Show that the function

$$f(z) = \begin{cases} \frac{\tan z}{z}, & z \neq 0,\ z \neq \pi/2 + k\pi,\ k \in \mathbb{Z} \\ 1, & z = 0 \end{cases}$$

is analytic in its domain.

[10 pts] **Problem 3.** (*No credit will be given for just stating the Laurent expansion without showing all required work. You can use without proof known expansions such as e^z etc.*)

(i) What type of singularity does the function

$$f(z) = \frac{1}{z^4 \sin z}$$

have at the point 0? Find the first 3 terms in the Laurent expansion of f that is valid in some annulus $0 < |z| < R$. What is the residue of f at 0?

(ii) Suppose that a function $g(z)$ has the Laurent expansion

$$g(z) = \frac{-1}{z^2} + \frac{2i}{z} + 1 + i + z + 3iz^2 + \dots$$

in an annulus $0 < |z| < R$. Find the value of the following limit, if it exists:

$$\lim_{z \to 0} g(z).$$

In case it does not exist, explain the reason.

[10 pts] **Problem 4.** (*No credit will be given for just stating the expansions without showing all required work. You can use without proof known expansions such as e^z etc.*)
Consider the function

$$g(z) = \frac{1}{1+z^2}.$$

Let $G(z)$ be the antiderivative of $g(z)$ defined in the the unit disk $|z| < 1$ such that $G(0) = 0$. Find the Maclaurin expansion of $G(z)$.

[10 pts] **Problem 5.**

(i) Let f be an analytic function in a domain D. State the Cauchy Integral Formula for f.

(ii) Let C be the positively oriented circle $|z| = 3$ and consider the function

$$g(z) = \int_C \frac{2w^2 - e^w}{w - z} dw.$$

Show that the function $g(z)$ is analytic when $|z| < 3$ and $|z| > 3$.

[10 pts] **Problem 6.**

(i) State Liouville's theorem.

(ii) Suppose that f is an entire function and v is the imaginary part of f. If v is bounded, then show that the function f is constant.

[10 pts] **Problem 7.**

(i) Determine the number of zeros, counting multiplicities, of the polynomial

$$z^6 - 5z^4 + z^3 - 2z = 0$$

inside the circle $|z| = 1$.

(ii) Consider the function

$$f(z) = \frac{(2z-1)^7}{z^3}$$

and denote by C the unit circle with the counter-clockwise orientation. How many times does the image of C under f wind around the origin and in what orientation?

[10 pts] **Problem 8.** Using the residue at infinity evaluate the integral

$$\int_C \frac{5z - z^2}{(z-5)(z-1)^2} dz,$$

where C is the positively oriented circle $|z| = 6$.

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[10 pts] **Problem 9.** Calculate the improper integral

$$\int_0^\infty \frac{x \sin 2x}{x^2+3} dx.$$

Justify carefully all steps.

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[15 pts] **Problem 10.**

[5 pts] (i) Give the formula in polar coordinates for the branch of $z^{-1/2}$ that is defined in the complement of the negative imaginary axis including the origin, so that $(-1)^{-1/2} = -i$. Using that branch, describe the largest domain in which the function

$$f(z) = \frac{z^{-1/2}}{z^2 + 1}$$

is analytic.

[10 pts] (ii) Calculate the improper integral

$$\int_0^\infty \frac{x^{-1/2}}{x^2 + 1} dx.$$

Justify carefully all steps.

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Extra page for **Problem 10** (ii).