MAT342 Homework 11 Solutions

Due Wednesday, May 1

1. Use residues to show that $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{2}$.

Observe that $f(x) = 1/(x^2+1)^2$ is even, so the Cauchy principal value and the improper integral coincide.

The function $f(z)$ has a pole of order two at i (and also at $-i$, which is irrelevant for our purposes), with residue $-i/4$. Let $\mathcal{C}_R$ be the positively oriented semicircle $|z| = R$ with $0 \le \arg z \le \pi$. Then by the Cauchy-Goursat theorem, for any $R > 1$

$$\int_{-R}^{R} \frac{dx}{(1+x^2)^2} + \int_{\mathcal{C}_R} \frac{dz}{(1+z^2)^2} = 2\pi i \operatorname*{Res}_{z=i} f(z) = \frac{\pi}{2} .$$

Furthermore, $|f(z)| \le 1/(R^2-1)^2$ for $z \in \mathcal{C}_R$, and so

$$\lim_{R\to\infty} \left| \int_{\mathcal{C}_R} \frac{dz}{(z^2+1)^2} \right| \le \lim_{R\to\infty} \frac{\pi R}{(R^2-1)^2} = 0 .$$

Thus

$$\frac{\pi}{2} = \lim_{R\to\infty} \left(\int_{-R}^{R} \frac{dx}{(1+x^2)^2} + \int_{\mathcal{C}_R} \frac{dz}{(1+z^2)^2} \right) = \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} + 0.$$

2. Using residues, show that $\displaystyle\int_{0}^{\infty} \frac{x^2\,dx}{(x^2+9)(x^2+4)^2} = \frac{\pi}{200}$.

The function $f(z) = \frac{z^2}{(z^2+9)(z^2+4)^2}$ has simple poles at $z = \pm 3i$ and poles of order 2 at $z = \pm 2i$. The poles in the upper half-plane have residues of $3i/50$ and $-13i/200$, respectively. Let $\mathcal{C}$ be the usual positively oriented contour consisting of the semi-circle of radius R around the origin followed by the segment of the real axis from $-R$ to R. For $R > 3$ we have

$$\int_{\mathcal{C}} f(z)\,dz = 2\pi i \left(\frac{3i}{50} - \frac{13i}{200} \right) = \frac{\pi}{100} .$$

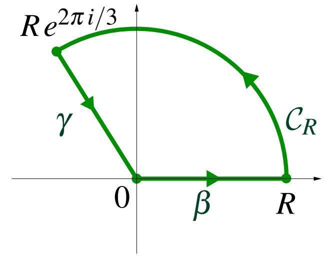
We need to confirm that the integral over the semicircle tends to 0 as $R \to \infty$, but since $|f(z)| < \frac{R^2}{(R^2-9)(R^2-4)^2}$, this follows readily.

Thus,

$$\frac{\pi}{100} = \lim_{R\to\infty} \int_{\mathcal{C}} f(z)\,dz = \lim_{R\to\infty} \left(\int\limits_{|z|=R} f(z)\,dz + \int_{-R}^{R} f(z)\,dz \right) = 0 + \int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+9)(x^2+4)^2} .$$

Since f is an even function, the integral from 0 to ∞ is half of the above, as desired.

3. Using a contour like the one at right with R sufficiently large, show that $\displaystyle\int_0^\infty \frac{dx}{x^3+8} = \frac{\pi}{6\sqrt{3}}$.



As in the previous problems, we express the desired integral as part of a larger contour which surrounds a singular point, and write $f(z) = 1/(z^3+8)$. Here the relevant singularity is the pole at $z = 2e^{\pi i/3}$ and

$$\operatorname*{Res}_{z=2e^{\pi i/3}} f(z) = \frac{1}{3(2e^{\pi i/3})^2} = \frac{1}{12\,e^{2\pi i/3}} = \frac{1}{-6+6i\sqrt{3}} \ .$$

(The other two poles are at -1 and $z = 2e^{-\pi i/3}$ and lie outside of the contour.)

As in the figure, let $\mathcal{C}_R$ be the part of the positively oriented circle with $0 \le \arg z \le 2\pi i/3$ and $|z| = R$ with $R > 2$, and let γ represent the angled leg of the sector, and β be the part along the real axis. Take the contour $\mathcal{C}$ to be $\mathcal{C}_R$ followed by γ followed by β; the Cachy-Goursat theorem gives

$$\text{(1)} \qquad \frac{2\pi i}{12\,e^{2\pi i/3}} = \int_{\mathcal{C}} f(z)\,dz = \int_{\mathcal{C}_R} f(z)\,dz + \int_\gamma f(z)\,dz + \int_\beta f(x)\,dx \ .$$

Since $|f(z)| \le \dfrac{1}{R^3-8}$ for $z \in \mathcal{C}_R$, it is easy to see that $\displaystyle\lim_{R\to\infty} \int_{\mathcal{C}_R} f(z)\,dz = 0$.

Observe that we can parameterize β as $z = x$ with $0 \le x \le R$, and $-\gamma$ as $z = xe^{2\pi i/3}$ with $0 \le x \le R$. Using this parameterization, we see that

$$\int_\gamma f(z)\,dz = -\int_{-\gamma} f(z)\,dz = -\int_0^R \frac{e^{2\pi i/3}\,dx}{(xe^{2\pi i/3})^3+1} = -e^{2\pi i/3}\int_0^R \frac{dx}{x^3+1} = -e^{2\pi i/3}\int_\beta f(x)\,dx \ ,$$

that is, $\displaystyle\int_\beta f(z)\,dz + \int_\gamma f(z)\,dz = \left(1 - e^{2\pi i/3}\right)\int_0^R f(x)\,dx$.

Combining this with eq. (1) and taking the limit as $R \to \infty$ gives

$$\frac{\pi i}{6e^{2\pi i/3}} = \lim_{R\to\infty}\left(1 - e^{2\pi i/3}\right)\int_0^R f(x)\,dx = \left(1 - e^{2\pi i/3}\right)\int_0^\infty f(x)\,dx \ .$$

Hence

$$\int_0^\infty f(x)\,dx = \frac{\pi i}{6\,e^{2\pi i/3}(1-e^{2\pi i/3})} = \frac{\pi i}{6(e^{2\pi i/3} - e^{4\pi i/3})} = \frac{\pi i}{6\left(\frac{-1+\sqrt{3}i}{2} - \frac{-1-\sqrt{3}i}{2}\right)} = \frac{\pi}{6\sqrt{3}} \ .$$

4. Using residues, show that $\displaystyle\int_{-\infty}^{\infty} \frac{\cos 5x}{x^2+1}\,dx = \frac{\pi}{e^5}$.

Let $f(x) = 1/(x^2+1)$ and we integrate $f(z)\,e^{5iz}$ over the contour $\mathcal{C}$ consisting of the segment from $-R$ to R along the real axis followed by the semicircle $\mathcal{C}_R$ of radius R in the upper half-plane (with $R > 1$).

Since $|f(z)| \to 0$ on $\mathcal{C}_R$ as $R \to \infty$, Jordan's Lemma tells us that $\displaystyle\lim_{R\to\infty}\int_{\mathcal{C}_R} f(z)e^{5iz}\,dz = 0$, so

$$\int_{\mathcal{C}} \frac{e^{5iz}\,dz}{z^2+1} = \int\limits_{z\in\mathbb{R}} \frac{\cos 5z + i\sin 5z}{z^2+1}\,dz$$

and we can find the desired integral by taking the real part of both sides.

The residue at $z=i$ of $e^{5iz}/(z^2+1)$ is just $\phi(i)$ where $\phi(z) = e^{5iz}/(z+i)$, so we get $-i/(2e^5)$. Hence we have

$$\int_{-\infty}^{\infty} \frac{\cos 5x}{x^2+1}\,dx = \operatorname{Re}\left(\int_{\mathcal{C}} \frac{e^{5iz}\,dz}{z^2+1}\right) = \operatorname{Re}\left(2\pi i \operatorname*{Res}_{z=i} \frac{e^{5iz}\,dz}{z^2+1}\right) = \frac{\pi}{e^5}\ .$$

5. Use residues to calculate $\displaystyle\int_0^{\infty} \frac{x^3 \sin x}{(x^2+1)(x^2+9)}\,dx$.

This is similar to the previous problem.

Specifically, let $f(x) = x^3/((x^2+1)(x^2+9))$ and integrate $f(z)\,e^{iz}$ over the contour $\mathcal{C}$ consisting of the segment from $-R$ to R along the real axis followed by the semicircle $\mathcal{C}_R$ of radius R in the upper half-plane (with $R > 3$).

The function $f(z)e^{iz}$ has simple poles at $z = \pm i$ and $z = \pm 3i$, and the two residues inside the contour $\mathcal{C}$ are $-1/(16e)$ and $9/(16e^3)$, respectively.

As in the previous problem, $|f(z)| \to 0$ on $\mathcal{C}_R$ as $R \to \infty$, and so Jordan's Lemma tells us that $\displaystyle\lim_{R\to\infty}\int_{\mathcal{C}_R} f(z)e^{iz}\,dz = 0$.

Putting this all together gives

$$\int_{-\infty}^{\infty} \frac{x^3\cos x\,dx}{(x^2+1)(x^2+9)} = \operatorname{Im}\left(\int_{\mathcal{C}} \frac{x^3 e^{iz}\,dz}{(z^2+1)(z^2+9)}\right) = \operatorname{Im}\left(2\pi i\left(\frac{9}{16e^3} - \frac{1}{16e}\right)\right) = \frac{(9-e^2)\pi}{8e^3}\ .$$

Since $f(x)\sin x$ is an even function, we have

$$\int_0^{\infty} \frac{x^3\cos x\,dx}{(x^2+1)(x^2+9)} = \frac{(9-e^2)\pi}{16e^3}\ .$$