MAT342 Homework 10
Due Wednesday, April 24

1. Let $\mathcal{C}$ be the positively oriented circle $|z| = 2$. Use residues to evaluate the integral of each of the following functions along $\mathcal{C}$.

(a) $\dfrac{e^{-z}}{z^2}$ (b) $\dfrac{e^{-z}}{(z-1)^2}$ (c) $z^2 e^{1/z}$

2. Let $f(z) = \dfrac{4z^2 - 5}{z(z-1)(1+z^2)}$, and let $\mathcal{C}$ be the positively oriented circle $|z| = 2$. Compute $\displaystyle\int_{\mathcal{C}} f(z)\,dz$. You can do this any number of ways, although I recommend calculating the residue at infinity.

3. Evaluate $\displaystyle\int_{\mathcal{C}} \frac{dz}{z^3(z+4)}$ where $\mathcal{C}$ is the positively oriented circle given by

(a) $|z| = 2$ (b) $|z+2| = 3$

4. Compute the residue at $z = 0$ for each of the following:

(a) $\dfrac{\sin z}{z^6}$ (b) $\dfrac{\cos z}{z^5}$ (c) e^{1/z^6} (d) $\dfrac{5z^3 - 4}{z(z-2)}$

5. Let $\mathcal{C}_N$ be the positively oriented boundary of the square joining the four points of the form $\pm(N+\frac{1}{2})\pi \pm i(N+\frac{1}{2})\pi$, with $N \in \mathbb{Z}^+$. Show that

$$\int_{\mathcal{C}_N} \frac{dz}{z^2 \sin z} = 2\pi i \left(\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2} \right).$$

One can show (but you don't have to — see §4.47 exercise 8) that $\displaystyle\left| \int_{\mathcal{C}_N} \frac{dz}{z^2 \sin z} \right| \le \frac{16}{(2N+1)\pi}$ and so the value of the integral over $\mathcal{C}_N$ tends to zero as $N \to \infty$.

Using this result, conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{25} + \ldots = \frac{\pi^2}{12}$.

6. Let $\mathcal{C}_R$ be the contour consisting of the segement of the real axis from $-R$ to R, and $\mathcal{C}_O$ be the semi-circular arc of radius R going from R back to $-R$ (in the upper half-plane); let $\mathcal{C}$ be the positively oriented contour consisting of $\mathcal{C}_R$ followed by $\mathcal{C}_O$.

(a) Compute $\displaystyle\int_{\mathcal{C}} \frac{dz}{1+z^4}$ for $R > 1$ (the value of the integral is zero for $R < 1$).

(b) Observe that for $z \in \mathcal{C}_O$, we know that $\frac{1}{R^4-1} \ge \left|\frac{1}{1+z^4}\right| \ge \frac{1}{R^4+1}$ (since $1+z^4$ is the distance in $\mathbb{C}$ between z^4 and -1). Use this to conclude that

$$\lim_{R\to\infty} \int_{\mathcal{C}_O} \frac{dz}{1+z^4} = 0.$$

(c) Finally, combine the results of the previous two parts to calculate

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} \quad \text{as} \quad \lim_{R\to\infty} \int_{\mathcal{C}_R} \frac{dz}{1+z^4} = \lim_{R\to\infty} \int_{\mathcal{C}} \frac{dz}{1+z^4}.$$

(The first limit is a valid represention of the integral since $1/(1+x^4)$ is a nonzero, even function of x. If the function were not even, we would have to find the integral for $x < 0$ and $x > 0$ separately.)