MAT342 Homework 10 Solutions

Due Wednesday, April 24

1. Let $\mathcal{C}$ be the positively oriented circle $|z| = 2$. Use residues to evaluate the integral of each of the following functions along $\mathcal{C}$.

(a) $\dfrac{e^{-z}}{z^2} \quad = 2\pi i \cdot \operatorname*{Res}_{z=0} \dfrac{e^{-z}}{z^2} = -2\pi i$

(b) $\dfrac{e^{-z}}{(z-1)^2} \quad = 2\pi i \cdot \operatorname*{Res}_{z=1} \dfrac{e^{-z}}{(z-1)^2} = \dfrac{-2\pi i}{e}$

(c) $z^2 e^{1/z} \quad = 2\pi i \cdot \operatorname*{Res}_{z=0} z^2 e^{1/z} = \dfrac{2\pi i}{6} = \dfrac{\pi i}{6}$

2. Let $f(z) = \dfrac{4z^2 - 5}{z(z-1)(1+z^2)}$, and let $\mathcal{C}$ be the positively oriented circle $|z| = 2$. Compute $\int_{\mathcal{C}} f(z)\,dz$.

You can do this any number of ways, although I recommend calculating the residue at infinity.

$f(z)$ has simple poles at $z = 0$, $z = 1$, $z = i$ and $z = -i$, all of which are inside $\mathcal{C}$. So, we can calculate the integral as $2\pi i$ times the sum of the residues or as $2\pi i$ times the residue at infinity. I'll do both.

$$\operatorname*{Res}_{z=\infty} f(z) = \operatorname*{Res}_{z=0} \frac{f(1/z)}{z^2} = \operatorname*{Res}_{z=0} \frac{5z^2 - 4}{(z-1)(z^2+1)} = 0\ .$$

Or, if you prefer,

$$\begin{aligned}
\operatorname*{Res}_{z=0} f(z) &= \operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=-i} f(z) \\
&= \frac{-5}{-1\cdot 1} + \frac{4-5}{1\cdot 2} + \frac{-4-5}{i\cdot(i-1)\cdot(i+i)} + \frac{-4-5}{-i\cdot(-i-1)\cdot(-i-i)} \\
&= 5 - \tfrac{1}{2} - \tfrac{9}{4}(1+i) - \tfrac{9}{4}(1-i) \qquad = 0\ .
\end{aligned}$$

If you are a masochist, you can parameterize $\mathcal{C}$ as $\theta \mapsto 2e^{i\theta}$ and do the integral

$$\int_0^{2\pi} \frac{16e^{2i\theta}}{2e^{i\theta}(2e^{i\theta}-1)(4e^{2i\theta}+1)} \cdot (2ie^{i\theta})\,d\theta\ ,$$

but that is just *horrible* and you get zero anyway.

3. Evaluate $\int_{\mathcal{C}} \dfrac{dz}{z^3(z+4)}$ where $\mathcal{C}$ is the positively oriented circle given by

(a) $|z| = 2$ $\quad$ $\mathcal{C}$ contains 0 but not -4, so we get $2\pi i \operatorname*{Res}_{z=0} f(z) = \dfrac{2\pi i}{64} = \dfrac{\pi i}{32}$.

(b) $|z+2| = 3$ $\quad$ $\mathcal{C}$ contains both -4 and 0. Since $\operatorname*{Res}_{z=-4} f(z) = -\dfrac{1}{64}$, the integral gives 0.

4. Compute the residue at $z=0$ for each of the following:

(a) $\dfrac{\sin z}{z^6} = \dfrac{1}{z^5} - \dfrac{1}{6z^3} + \dfrac{1}{120z} + \mathcal{O}(z)$, so the residue is $\dfrac{1}{120}$.

(b) $\dfrac{\cos z}{z^5} = \dfrac{1}{z^5} - \dfrac{1}{2z^3} + \dfrac{1}{24z} + \mathcal{O}(z)$, so the residue is $\dfrac{1}{24}$.

(c) $e^{1/z^6} = 1 + \dfrac{1}{z^6} + \mathcal{O}(z^{-12})$, so the residue is 0 .

(d) $\dfrac{5z^3-4}{z(z-2)}$ The residue is $\dfrac{5\cdot 0-4}{-2} = 2$.

5. Let $\mathcal{C}_N$ be the positively oriented boundary of the square joining the four points of the form $\pm(N+\frac{1}{2})\pi \pm i(N+\frac{1}{2})\pi$, with $N \in \mathbb{Z}^+$. Show that

$$\int_{\mathcal{C}_N} \frac{dz}{z^2 \sin z} = 2\pi i \left(\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2} \right).$$

One can show (but you don't have to — see §4.47 exercise 8) that $\left| \int_{\mathcal{C}_N} \frac{dz}{z^2 \sin z} \right| \le \frac{16}{(2N+1)\pi}$ and so the value of the integral over $\mathcal{C}_N$ tends to zero as $N \to \infty$.

Using this result, conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{25} + \ldots = \frac{\pi^2}{12}$.

For notation, let $f(z) = \frac{1}{z^2 \sin z}$. The pole at $z=0$ is not simple, but the Laurent series gives $f(z) = \frac{1}{z^3} + \frac{1}{6z} + \mathcal{O}z$, so the residue at $z=0$ is $\frac{1}{6}$.

The remaining zeros of $z^2 \sin z$ occur at $\pm n\pi$ for $n \in \mathbb{Z}^+$, the corresponding poles are all simple, and for $n > 0$

$$\operatorname*{Res}_{z=\pm n\pi} \frac{1}{z^2 \sin z} = \frac{(-1)^n}{(n\pi)^2}.$$

To see this, observe the for n odd the first term in the Taylor series for $\sin z$ near $z = n\pi$ is $-(z-n\pi)$, but for n even the first term is $(z-n\pi)$. Thus we can write

$$\frac{1}{z^2 \sin z} = \frac{(-1)^n/z^2}{(z-n\pi) + \mathcal{O}\big((z-n\pi)^3\big)}$$

and conclude that the residue at $z = \pm n\pi$ is $(-1)^n/(n\pi)^2$. Using this we have

$$\int_{\mathcal{C}_N} \frac{dz}{z^2 \sin z} = 2\pi i \left(\operatorname*{Res}_{z=0} f(z) + \sum_{n=1}^{N} \left(\operatorname*{Res}_{z=n\pi} f(z) + \operatorname*{Res}_{z=-n\pi} f(z) \right) \right) = 2\pi i \left(\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2} \right).$$

Now, using $\left| \int_{\mathcal{C}_N} \frac{dz}{z^2 \sin z} \right| \le \frac{16}{(2N+1)\pi}$ (which follows from the fact that $|\sin z| \ge 1$ for $z \in \mathcal{C}_N$), we can see that $\lim_{N\to\infty} \left| \int_{\mathcal{C}_N} f(z)\,dz \right| = 0$. Hence

$$0 = \int_{\mathcal{C}_N} \frac{dz}{z^2 \sin z} = 2\pi i \left(\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2} \right) \quad \text{so} \quad -\frac{1}{6} = 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}.$$

Multiplying both sides by $-\frac{\pi^2}{2}$ gives the result.

6. Let $\mathcal{C}_R$ be the contour consisting of the segement of the real axis from $-R$ to R, and $\mathcal{C}_O$ be the semi-circular arc of radius R going from R back to $-R$ (in the upper half-plane); let $\mathcal{C}$ be the positively oriented contour consisting of $\mathcal{C}_R$ followed by $\mathcal{C}_O$.

(a) Compute $\int_{\mathcal{C}} \frac{dz}{1+z^4}$ for $R > 1$ (the value of the integral is zero for $R < 1$).

Note that $f(z) = 1/(1+z^4)$ has poles at the fourth roots of -1. Let $p_+ = (1+i)/\sqrt{2}$ and $p_- = (1-i)/\sqrt{2}$ be the two of these which lie inside the contour $\mathcal{C}$. Then

$$\int_{\mathcal{C}} \frac{dz}{1+z^4} = 2\pi i\Big(\operatorname*{Res}_{z=p_+} f(z) + \operatorname*{Res}_{z=p_-} f(z)\Big) = 2\pi i \left(\frac{-1-i}{4\sqrt{2}} + \frac{1-i}{4\sqrt{2}} \right) = \frac{\pi}{\sqrt{2}} \ .$$

(b) Observe that for $z \in \mathcal{C}_O$, we know that $\frac{1}{R^4-1} \geq \left| \frac{1}{1+z^4} \right| \geq \frac{1}{R^4+1}$ (since $1+z^4$ is the distance in $\mathbb{C}$ between z^4 and -1). Use this to conclude that

$$\lim_{R\to\infty} \int_{\mathcal{C}_O} \frac{dz}{1+z^4} = 0 \ .$$

As suggested, for $z \in \mathcal{C}_O$ we have $R^4+1 \leq \left|1+z^4\right| \leq R^4-1$ by interpreting $\left|1+z^4\right|$ as the distance between z^4 and -1 and observing that the image of $\mathcal{C}_O$ under $z \mapsto z^4$ is the circle of radius R^4 (covered twice). (If you prefer, you can instead observe that the image of $\mathcal{C}_O$ under $z \mapsto 1+z^4$ is the circle of radius R^4 and center 1 to get the same result.) Consequently, $|f(z)| \leq 1/(1+R^4)$ for all $z \in \mathcal{C}_O$. Furthermore, the length of $\mathcal{C}_O$ is πR. Using the bound on the modulus of the integral (§4.47), we have

$$\left| \int_{\mathcal{C}_O} \frac{dz}{1+z^4} \right| \leq \frac{1}{1+R^4} \cdot \pi R \ ,$$

which tends to 0 as $R \to \infty$.

(c) Finally, combine the results of the previous two parts to calculate

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} \quad \text{as} \quad \lim_{R\to\infty} \int_{\mathcal{C}_R} \frac{dz}{1+z^4} \ = \ \lim_{R\to\infty} \int_{\mathcal{C}} \frac{dz}{1+z^4} \ .$$

(The first limit is a valid represention of the integral since $1/(1+x^4)$ is a nonzero, even function of x. If the function were not even, we would have to find the integral for $x < 0$ and $x > 0$ separately.)

Since $\int_{\mathcal{C}} \frac{dz}{1+z^4} = \frac{\pi}{\sqrt{2}}$ for all $R > 1$, we have

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} \ = \ \lim_{R\to\infty} \int_{\mathcal{C}_R} \frac{dz}{1+z^4} + 0 \ = \ \lim_{R\to\infty} \int_{\mathcal{C}_R} \frac{dz}{1+z^4} + \lim_{R\to\infty} \int_{\mathcal{C}_O} \frac{dz}{1+z^4} \ = \ \int_{\mathcal{C}} \frac{dz}{1+z^4} = \frac{\pi}{\sqrt{2}} \ .$$