MAT342 Homework 9

Due Wednesday, April 17

1. Give two Laurent series expansions (of the form $\sum_{n=-\infty}^{\infty} c_n z^n$) for $f(z) = \dfrac{1}{z^3(1-z^2)}$ and state the regions on which the expansions are valid. (Hint: I find it useful to write f as a sum of fractions.)

2. By examining the Maclaurin series for $\frac{1-\cos z}{z^2}$, show that the function

$$f(z) = \begin{cases} \dfrac{1-\cos z}{z^2} & \text{for } z \neq 0, \\ \frac{1}{2} & \text{for } z = 0 \end{cases}$$

is entire.

3. (a) The Taylor series for $\frac{1}{w}$ about the point $w = 1$ is given by

$$\frac{1}{w} = \sum_{n=0}^{\infty} (-1)^n (w-1)^n \qquad \text{for } |w-1| < 1,$$

which can easily be seen by substituting $1 - z = w$ into the geometric series. Integrate the above series along a contour lying inside the disk of convergence from $w = 1$ to $w = z$ and obtain the series for the principal value of the logarithm

$$\operatorname{Log} z = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(z-1)^n}{n} \qquad \text{for } |z-1| < 1.$$

(b) Using the previous part, show that the function

$$f(z) = \begin{cases} \dfrac{\operatorname{Log} z}{z-1} & \text{for } z \neq 1, \\ 1 & \text{for } z = 1 \end{cases}$$

is analytic throughout the slit plane $z \neq 0,\ -\pi < \operatorname{Arg} z < \pi$.

4. Multiply the Maclaurin series for e^z and $\frac{1}{1+z}$ to obtain the series expansion for $\dfrac{e^z}{1+z}$ up to z^5. On what disk does it converge?

5. Use division of power series to obtain the first three nonzero terms of the Laurent series for $\dfrac{1}{\sinh z}$ valid for $0 < |z| < \pi$.