MAT342 Homework 9 Solutions

Due Wednesday, April 17

1. Give two Laurent series expansions (of the form $\sum_{n=-\infty}^{\infty} c_n z^n$) for $f(z) = \dfrac{1}{z^3(1-z^2)}$ and state the regions on which the expansions are valid. (Hint: I find it useful to write f as a sum of fractions.)

As suggested, write $f(z) = \dfrac{1}{z^3} + \dfrac{1}{z} - \dfrac{z}{1-z^2}$. Observe that the terms $\frac{1}{z^3}$ and $\frac{1}{z}$ are analytic for $z \neq 0$, while $\frac{z}{1-z^2}$ has singlarities at ± 1.

From the expansion for a geometric series, we obtain

$$\frac{z}{1-z^2} = z\sum_{n=0}^{\infty}(z^2)^n = \sum_{n=0}^{\infty} z^{2n+1} \qquad \text{for } |z| < 1,$$

and so

$$f(z) = \frac{1}{z^3} + \frac{1}{z} - \sum_{n=0}^{\infty} z^{2n+1} = \frac{1}{z^3} + \frac{1}{z} - z - z^3 - z^5 - \dots \qquad \text{for } 0 < |z| < 1.$$

If $|z| > 1$, then the we can express $\frac{z}{1-z^2}$ as a geometric series in $\frac{1}{z}$. That is,

$$\frac{z}{1-z^2} = \frac{1/z}{1/z^2 - 1} = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z^2}\right)^n = -\sum_{n=0}^{\infty}\frac{1}{z^{2n+1}} = -\frac{1}{z} - \frac{1}{z^3} - \frac{1}{z^5} - \dots \qquad \text{for } 1 < |z|.$$

Consequently,

$$f(z) = \frac{1}{z^3} + \frac{1}{z} + \sum_{n=0}^{\infty}\frac{1}{z^{2n+1}} = \frac{2}{z} + \frac{2}{z^3} + \sum_{n=2}^{\infty}\frac{1}{z^{2n+1}} = \frac{2}{z} + \frac{2}{z^3} + \frac{1}{z^5} + \frac{1}{z^7} + \dots \qquad \text{for } 1 < |z|.$$

2. By examining the Maclaurin series for $\frac{1-\cos z}{z^2}$, show that the function

$$f(z) = \begin{cases} \dfrac{1-\cos z}{z^2} & \text{for } z \neq 0, \\ \frac{1}{2} & \text{for } z = 0 \end{cases}$$

is entire.

Writing the series for $\frac{1-\cos z}{z^2}$, we have

$$\frac{1}{z^2}\Big(1 - \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}\Big) = \frac{1}{z^2}\cdot\sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^{2n}}{(2n)!} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^{2n-2}}{(2n)!} = \frac{1}{2} - \frac{z^2}{4!} + \dots$$

Since for $z \neq 0$, $f(z) = \frac{1-\cos z}{z^2}$ is analytic and agrees with the series expansion. But the series is convergent for all z and takes on the value $1/2 = f(0)$ at $z = 0$. Hence the singularity of $\frac{1-\cos z}{z^2}$ is removable, and f is an entire function.

3. (a) The Taylor series for $\frac{1}{w}$ about the point $w = 1$ is given by

$$\frac{1}{w} = \sum_{n=0}^{\infty} (-1)^n (w-1)^n \qquad \text{for } |w-1| < 1,$$

which can easily be seen by substituting $1 - z = w$ into the geometric series. Integrate the above series along a contour lying inside the disk of convergence from $w = 1$ to $w = z$ and obtain the series for the principal value of the logarithm

$$\operatorname{Log} z = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(z-1)^n}{n} \qquad \text{for } |z-1| < 1.$$

Let γ be any contour lying in the disk $\{ w \mid |w-1| < 1 \}$ with $\gamma 0 = 1$ and $\gamma 1 = z$. Since $1/w$ is analytic in this disk, it has an antiderivative $\operatorname{Log} w$ and so

$$\int_\gamma \frac{dw}{w} = \operatorname{Log} z - \operatorname{Log} 1 = \operatorname{Log} z \ .$$

Since the series converges in the disk, we can integrate term-by-term to obtain

$$\operatorname{Log} z = \int_\gamma \frac{dw}{w} = \int_\gamma \sum_{n=0}^{\infty} (-1)^n (w-1)^n \, dw = \sum_{n=0}^{\infty} \frac{(-1)^n (w-1)^{n+1}}{n+1} \Bigg|_1^z = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(z-1)^n}{n} \ ,$$

reindexing the series after evaluating the integral.

(b) Using the previous part, show that the function

$$f(z) = \begin{cases} \dfrac{\operatorname{Log} z}{z-1} & \text{for } z \neq 1, \\ 1 & \text{for } z = 1 \end{cases}$$

is analytic throughout the slit plane $z \neq 0$, $-\pi < \operatorname{Arg} z < \pi$.

Dividing the series in the previous part by $z - 1$, we get the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(z-1)^{n-1}}{n} = 1 - \frac{z-1}{2} + \dots \ ,$$

which converges in the disk $|z-1| < 1$. As in the previous problem, this means the singularity of $\operatorname{Log} z/(z-1)$ at $z = 1$ is removable, so $f(z)$ is analytic throughout the disk.

Note, however, the function $\operatorname{Log} z$ is analytic in the slit plane $\{ z \mid z \neq 0 \text{ and } -\pi < \operatorname{Arg} z < \pi \}$, and $z - 1 \neq 0$ for $z \neq 1$. Hence $f(z) = \operatorname{Log} z/(z-1)$ is analytic *outside* the disk as well (except on the slit), even though the given series representation does not converge there. Thus $f(z)$ is analytic everywhere except on the negative real axis or zero.

If you like, you could describe this in terms of integrals on arbitrary contours around $z = 1$ that avoid the negative real axis. It is effectively the same thing.

4. Multiply the Maclaurin series for e^z and $\frac{1}{1+z}$ to obtain the series expansion for $\dfrac{e^z}{1+z}$ up to z^5. On what disk does it converge?

$$
\begin{aligned}
e^z \cdot \frac{1}{1+z} &= \left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\frac{z^5}{5!}+\mathcal{O}(z^6)\right)\left(1-z+z^2-z^3+z^4-z^5+\mathcal{O}(z^6)\right) \\
&= 1+(-1+1)z+\left(1-1+\frac{1}{2!}\right)z^2+\left(-1+1-\frac{1}{3!}+\frac{1}{4!}\right)z^3+\left(1-1+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right)z^4 \\
&\qquad +\left(-1+1-\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\frac{1}{5!}\right)z^5+\mathcal{O}(z^6) \\
&= 1+\frac{z^2}{2}-\frac{z^3}{3}+\frac{3z^4}{8}-\frac{11z^5}{30}+O(z^6)\ .
\end{aligned}
$$

Since e^z is entire and the series for $\frac{1}{1+z}$ converges for $|z|<1$, the resulting series converges on the disk $|z|<1$.

5. Use division of power series to obtain the first three nonzero terms of the Laurent series for $\dfrac{1}{\sinh z}$ valid for $0<|z|<\pi$.

Note that $\sinh(z) = z+\frac{z^3}{3!}+\frac{z^5}{5!}+\mathcal{O}(z^7)$. Doing long division, we have

$$
\begin{array}{r|l}
 & 1/z-\frac{1}{6}z+\frac{7}{360}z^3+\mathcal{O}(z^5) \\ \hline
z+\frac{z^3}{6}+\frac{z^5}{120} & 1 \\
 & 1+\frac{z^2}{6}+\frac{z^4}{120} \\ \hline
 & -\frac{z^2}{6}-\frac{z^4}{120} \\
 & -\frac{z^2}{6}-\frac{z^4}{36}+\frac{z^6}{720} \\ \hline
 & \frac{7}{360}z^4+\mathcal{O}(z^6) \\
 & \frac{7}{360}z^4+\mathcal{O}(z^6) \\ \hline
 & \mathcal{O}(z^6)
\end{array}
$$

Note that we can stop the process at this stage since we only want the first three nonzero terms. Thus we have $\dfrac{1}{\sinh z} = \dfrac{1}{z}-\dfrac{z}{6}+\dfrac{7}{360}z^3+\mathcal{O}(z^5)$.

The three zeros of $\sinh z$ of smallest magnitude occur at $z=-i\pi$, $z=0$, and $z=i\pi$, so the above Laurent series is valid for $0<|z|<\pi$.