MAT342 Homework 8 Solutions
Due Wednesday, April 10

1. (a) Let f be continuous on a closed, bounded region R, and analytic on the interior of R. If f is not constant and nonzero throughout R, prove that $|f(z)|$ has a *minimum* value which occurs on the boundary of R and never in the interior.
Hint: apply the maximum principle to $g(z) = 1/f(z)$.

Since $f(z) \neq 0$ throughout R and f is analytic inside R, $g(z) = 1/f(z)$ is also analytic on the interior of R. Hence, by the maximum principle, $|g(z)|$ attains its maximum on the boundary of R. But the maximum of $|g|$ is a minumum of $|f|$, so $|f|$ attains both its maximum and minimum on the boundary of R, and all values on the interior of R must be between these.

(b) Show that the condition $f(z) \neq 0$ is necessary in the previous part. That is, give an example of a nonconstant, analytic function $f(z)$ and a closed bounded region R where $|f(z)|$ has a minimum interior to R (which is smaller than at any point on the boundary).
Hint: the identity function $f(z) = z$ will work.

Let R be the closed unit disk $\mathbb{D} = \{ z \mid |z| \leq 1 \}$, and consider $f(z) = z$. This function is analytic throughout $\mathbb{D}$, but $|f(0)| = 0$ and $|f(z)| = 1$ on the boundary of $\mathbb{D}$.

2. Let R be the rectangular region $0 \leq \operatorname{Re} z \leq 1$, and $0 \leq \operatorname{Im} z \leq \pi$. Determine the values z where the norm of the function e^z attains its maximum and minimum values.

From the maximum principle and the previous problem, we know both the maximum and minimum of $|e^z|$ occur on the boundary of R since $f(z) = e^z$ is never zero for any z. Writing $z = x + iy = (x, y)$, we must consider $|e^z|$ for points of the form $(x, 0)$, $(0, y)$, $(1, y)$, and (x, π) with $0 \leq x \leq 1$ and $0 \leq y \leq \pi$.

For $z = x + 0i$, e^x is an increasing function, so the minimum of 1 occurs at $(0, 0)$, and the maximum of e occurs at $(1, 0)$. For $z = 0 + iy$, e^{iy} is a point on the unit circle, so $|e^z| = 1$ for all such z. Similarly, when $z = 1 + iy$, e^{1+iy} is a point on the circle of radius e, so $|e^z| = e$ throughout. Finally, for $z = x + i\pi$, $e^z = -e^x$, so $|e^z|$ attains its minimum value of 1 at $z = \pi i$, and its maximim value of e at $z = 1 + \pi i$.

In short, the minimum of $|e^z|$ on the rectangle R is 1, which occurs at every point on the left boundary (where $\operatorname{Re} z = 0$) and the maximum of $|e^z|$ is e and occurs on the right boundary (where $\operatorname{Re} z = 1$).

Of course, the easy way to do this is to realize that e^z sends horizontal lines to rays through the origin, and vertical lines to concentric circles, so the maximum of $|e^z|$ must occur at every point on the entire vertical line of largest real part, and the minumum on the vertical line of smallest real part.

3. Using the definition of the limit of a sequence, show that $\lim_{n \to \infty} \left(2i + \frac{i^n}{n^2} \right) = 2i$.

Given $\varepsilon > 0$, we need to find N such that for all $n > N$, we have $\left| \left(2i + \frac{i^n}{n^2} \right) - 2i \right| < \varepsilon$.

Suppose that $N \geq \sqrt{1/\varepsilon}$. Then for any $n > N$, we have

$$\left| \left(2i + \frac{i^n}{n^2} \right) - 2i \right| < \left| \frac{i^N}{N^2} \right| < \frac{1}{1/(\sqrt{\varepsilon})^2} = \frac{1}{1/\varepsilon} = \varepsilon \ ,$$

as desired.

4. Obtain the Maclaurin series $z\cosh(z^2) = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!}$. For what z does it converge?

Recall that $\cosh w = \sum_{n=0}^{\infty} \frac{w^{2n}}{(2n)!}$ for all $w \in \mathbb{C}$, and so (letting $w = z^2$) we have

$$z\cosh z^2 = z \cdot \sum_{n=0}^{\infty} \frac{(z^2)^{2n}}{(2n)!} = z \cdot \sum_{n=0}^{\infty} \frac{z^{4n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!} .$$

Since $\cosh z$, z, and z^2 are all entire functions, this series converges for all $z \in \mathbb{C}$.

5. Using the fact that $e^z = e \cdot e^{z-1}$, obtain the Taylor series $e^z = e \cdot \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$.

Using $e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!}$ and letting $w = z-1$, we have $e^z = e \cdot e^{z-1} = e \cdot \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$.

6. Let $f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$. Write the Laurent series for $f(z)$ when $1 < |z| < 2$.
Hint: rewrite the first term of f in terms of $\frac{1}{1-1/z}$ and the second using $\frac{1}{1-(z/2)}$, then use geometric series.

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1-1/z} - \frac{1}{2} \cdot \frac{1}{1-z/2} = \frac{1}{z} \cdot \sum_{n=0}^{\infty} \frac{1}{z^n} - \frac{1}{2} \cdot \sum_{n=0}^{\infty} \frac{z^n}{2^n} = \sum_{n=1}^{\infty} \frac{1}{z^n} - \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} .$$

The series $\sum \frac{1}{z^n}$ converges for $|z| > 1$, and the series $\sum \frac{z^n}{2^n}$ converges for $|z| < 2$, so their difference converges for $1 < |z| < 2$.