MAT342 Homework 7

Due Wednesday, April 3

1. Let $f(z) = \dfrac{2z}{z^2-1} = \dfrac{1}{z+1} + \dfrac{1}{z-1}$.

(a) Let $\mathcal{C}_5$ be the contour consisting of the positively oriented circle of radius 1 centered at $z_0 = 5$, and compute $\displaystyle\int_{\mathcal{C}_5} f(z)\,dz$. Hint: it is easier to do this without parameterizing $\mathcal{C}_5$.

(b) Let $\mathcal{C}_1$ be the contour consisting of the positively oriented circle of radius 1 centered at $z_0 = 1$, and compute $\displaystyle\int_{\mathcal{C}_1} f(z)\,dz$. Hint: again, don't parameterize $\mathcal{C}_1$.

(c) Let $\mathcal{C}$ be the positively oriented circle of radius 3 centered at the origin. Compute $\displaystyle\int_{\mathcal{C}} f(z)\,dz$. Hint: Observe (without calculating) that the integral over the circle of radius 1 around $z_0 = -1$ has same value as in (b), and use the result of (b).

2. (a) Let $\mathcal{C}$ be any positively oriented simple closed contour, and let $\mathcal{R}$ be the region enclosed by $\mathcal{C}$. Use Green's Theorem (from multivariable calculus, or see section 4.50 of the text) to show that the area of $\mathcal{R}$ is given by

$$\text{Area}(\mathcal{R}) = \frac{1}{2i}\int_{\mathcal{C}} \overline{z}\,dz$$

even though the function $f(z)$ is nowhere analytic.

(b) Give an example of a closed contour $\mathcal{B}$ where $\dfrac{1}{2i}\displaystyle\int_{\mathcal{B}} \overline{z}\,dz$ does *not* equal the area enclosed by $\mathcal{B}$. Hint: Find a closed contour $\mathcal{B}$ which intersects itself once and the value of the integral is 0.

3. Let $\mathcal{C}$ be the circle $|z| = 4$, oriented counterclockwise, and define $g(z) = \displaystyle\int_{\mathcal{C}} \frac{s^3+1}{s-z}\,ds$ for $|z| \neq 4$.

(a) Calculate $g(0)$. (b) Calculate $g(2i)$. (c) Calculate $g(5)$.

4. Let $\mathcal{C}$ be the circle $|z| = 4$, oriented counterclockwise, and define $h(z) = \displaystyle\int_{\mathcal{C}} \frac{s^3+1}{(s-z)^3}\,ds$ for $|z| \neq 4$.

(a) Calculate $h(0)$. (b) Calculate $h(2i)$. (c) Calculate $h(5)$.

5. Suppose f is analytic within and on a simple closed contour $\mathcal{C}$, and z_0 is not on $\mathcal{C}$. Show that

$$\int_{\mathcal{C}} \frac{f'(z)\,dz}{z-z_0} = \int_{\mathcal{C}} \frac{f(z)\,dz}{(z-z_0)^2}\,.$$

6. Let f be an entire function with $|f(z)| \leq 2|z|$ for all $z \in \mathbb{C}$, and that $f(1) = 1$. Find $f(z)$.
Hint: Use Cauchy's inequality to show $f''(z) = 0$. This, together with the observation that $f(0) = 0$, should tell you $f(z)$.