MAT342 Homework 7 Solutions
Due Wednesday, April 3

1. Let $f(z) = \dfrac{2z}{z^2-1} = \dfrac{1}{z+1} + \dfrac{1}{z-1}$.

(a) Let $\mathcal{C}_5$ be the contour consisting of the positively oriented circle of radius 1 centered at $z_0 = 5$, and compute $\int_{\mathcal{C}_5} f(z)\,dz$. Hint: it is easier to do this without parameterizing $\mathcal{C}_5$.

Since $f(z)$ is analytic on and inside $\mathcal{C}_5$, the integral is zero by the Cauchy-Goursat Theorem.

(b) Let $\mathcal{C}_1$ be the contour consisting of the positively oriented circle of radius 1 centered at $z_0 = 1$, and compute $\int_{\mathcal{C}_1} f(z)\,dz$. Hint: again, don't parameterize $\mathcal{C}_1$.

Observe that $\displaystyle\int_{\mathcal{C}_1} f(z)\,dz = \int_{\mathcal{C}_1} \frac{dz}{z+1} + \int_{\mathcal{C}_1} \frac{dz}{z-1} = 0 + 2\pi i = 2\pi i.$
The first integral is zero because $\frac{1}{z+1}$ is anaytic on and inside $\mathcal{C}_1$; the second has $\log z$ as an antiderivative on the complex plane slit from 1 to ∞ (and, after a change of variables, is an integral we've done many times already).

(c) Let $\mathcal{C}$ be the positively oriented circle of radius 3 centered at the origin. Compute $\int_{\mathcal{C}} f(z)\,dz$.
Hint: Observe (without calculating) that the integral over the circle of radius 1 around $z_0 = -1$ has same value as in (b), and use the result of (b).

Let $\mathcal{C}_{-1}$ denote the positively oriented circle of radius 1 around -1, Since f is analytic on the region between $\mathcal{C}_3$ and $\mathcal{C}_1 \cup \mathcal{C}_{-1}$ (as well as on these contours), we have

$$\int_{\mathcal{C}_3} f(z)\,dz = \int_{\mathcal{C}_{-1}} f(z)\,dz + \int_{\mathcal{C}_{-1}} f(z)\,dz = 2\pi i + 2\pi i = 4\pi i\ .$$

2. (a) Let $\mathcal{C}$ be any positively oriented simple closed contour, and let $\mathcal{R}$ be the region enclosed by $\mathcal{C}$. Use Green's Theorem (from multivariable calculus, or see section 4.50 of the text) to show that the area of $\mathcal{R}$ is given by

$$\text{Area}(\mathcal{R}) = \frac{1}{2i}\int_{\mathcal{C}} \bar{z}\,dz$$

even though the function $f(z)$ is nowhere analytic.

Let $\mathcal{U}$ be the interior of $\mathcal{C}$. Recall that for functions P and Q from $\mathbb{R}^2$ to itself, Green's Theorem says that $\int_{\mathcal{C}} P\,dx + Q\,dy = \iint_{\mathcal{R}} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$. If $z = x+iy$ then $\bar{z} = x - iy$, so

$$\begin{aligned}\int_{\mathcal{C}} \bar{z}\,dz = \int_{\mathcal{C}} (x-iy)(dx+i\,dy) &= \int_{\mathcal{C}} (x\,dx + y\,dy) + i\int_{\mathcal{C}} (-y\,dx + x\,dy) \\ &= \iint_{\mathcal{R}} (0+0)\,dx\,dy + i\iint_{\mathcal{R}} (1+1)\,dx\,dy = 2i\iint_{\mathcal{R}} dx\,dy \\ &= 2i\,\text{Area}(\mathcal{R}).\end{aligned}$$

The result follows immediately.

(b) Give an example of a closed contour $\mathcal{B}$ where $\frac{1}{2i}\int_{\mathcal{B}} \overline{z}\,dz$ does *not* equal the area enclosed by $\mathcal{B}$. Hint: Find a closed contour $\mathcal{B}$ which intersects itself once and the value of the integral is 0.

Let $\mathcal{C}_0$ be the positively oriented circle of radius 1 and center 0, and let $\mathcal{C}_2$ be the positively oriented circle of radius 1 and center 2. Now let $\mathcal{B}$ be the contour $\mathcal{C}_0 - \mathcal{C}_2$, that is, the figure eight taken by starting at 1, going around the origin in a counter-clockwise way, then going around 2 clockwise. Since

$$\int_{\mathcal{B}} \overline{z}dz \;=\; \int_{\mathcal{C}_0} \overline{z}dz + \int_{-\mathcal{C}_2} \overline{z}dz \;=\; \int_{\mathcal{C}_0} \overline{z}dz - \int_{\mathcal{C}_2} \overline{z}dz \;=\; \pi - \pi \;=\; 0$$

but the area inside $\mathcal{B}$ is 2π, not $\frac{0}{2i}$.

3. Let $\mathcal{C}$ be the circle $|z| = 4$, oriented counterclockwise, and define $g(z) = \int_{\mathcal{C}} \frac{s^3+1}{s-z}\,ds$ for $|z| \neq 4$.

We use the Cauchy integral formula for the first two parts, so $g(z) = \int_{\mathcal{C}} f(s)/(s-z)\,ds$ with $f(s) = s^3+1$, which is entire, so if z is inside $\mathcal{C}$, $g(z) = 2\pi i\, f(z)$.

(a) Calculate $g(0)$. Since $f(0) = 1$, $g(0) = 2\pi i$.

(b) Calculate $g(2i)$. Since $f(2i) = -8i + 1$, $g(0) = 16\pi + 2\pi i$.

(c) Calculate $g(5)$. Note that $f(s)/(s-5)$ is analytic for s inside $\mathcal{C}$, so $g(0) = 0$.

4. Let $\mathcal{C}$ be the circle $|z| = 4$, oriented counterclockwise, and define $h(z) = \int_{\mathcal{C}} \frac{s^3+1}{(s-z)^3}\,ds$ for $|z| \neq 4$.

We use the extension of the Cauchy integral formula for the first two parts, and $h(z) = \int_{\mathcal{C}} f(s)/(s-z)^3\,ds$ with $f(s) = s^3+1$. Then for z inside $\mathcal{C}$, $h(z) = \pi i\, f''(z) = 6\pi i z$.

(a) Calculate $h(0)$. Since $f''(0) = 0$, $h(0) = 0$.

(b) Calculate $h(2i)$. Since $f''(2i) = 12i$, $h(2i) = -12\pi$.

(c) Calculate $h(5)$. Since $f(s)/(s-5)^3$ is analytic for s inside $\mathcal{C}$, $h(5) = 0$.

5. Suppose f is analytic within and on a simple closed contour $\mathcal{C}$, and z_0 is not on $\mathcal{C}$. Show that

$$\int_{\mathcal{C}} \frac{f'(z)\,dz}{z-z_0} = \int_{\mathcal{C}} \frac{f(z)\,dz}{(z-z_0)^2}\,.$$

Let $g(z) = f'(z)$, and assume $\mathcal{C}$ is positively oriented. Then g is also analytic in and on $\mathcal{C}$. If z_0 is exterior to $\mathcal{C}$, then both integrals are zero (since both integrands are analytic on the interior of $\mathcal{C}$) and hence equal.

If z_0 is inside $\mathcal{C}$, then by the extended Cauchy integral formula,

$$\int_{\mathcal{C}} \frac{f'(z)\,dz}{z-z_0} = \int_{\mathcal{C}} \frac{g(z)\,dz}{z-z_0} = 2\pi i\, g(z_0) = 2\pi i\, f'(z_0) = \int_{\mathcal{C}} \frac{f(z)\,dz}{(z-z_0)^2}\,.$$

If $\mathcal{C}$ were negatively oriented, apply the above argument to $-\mathcal{C}$ to get equality; then the original integrals are also equal since their negatives are equal.

6. Let f be an entire function with $|f(z)| \leq 2|z|$ for all $z \in \mathbb{C}$, and that $f(1) = 1$. Find $f(z)$.
Hint: Use Cauchy's inequality to show $f''(z) = 0$. This, together with the observation that $f(0) = 0$, should tell you $f(z)$.

Cauchy's inequality (p. 170 of the text) says that if f is analytic in and on a circle of radius R centered at z_0 and $|f(z)| \leq M$ on this circle, then $\left|f^{(n)}(z_0)\right| \leq \dfrac{n!M}{R^n}$. In our case $z_0 = 0$, and taking $n = 2$ and $|z| = R$, we have

$$\left|f''(0)\right| \leq \frac{2 \cdot 2R}{R^2} = \frac{4}{R}.$$

Since f is entire, this holds for *every* positive R, and hence we have $|f''(0)| \leq \varepsilon$ for every $\varepsilon > 0$, that is, $|f''(0)| = 0$.

Since $f''(z) = 0$, so $f'(z) = a$ for some constant a, and hence $f(z) = az + b$ for some b. But the assumption $|f(z)| \leq 2|z|$ tells us that $f(0) = 0$ (since it holds for $z = 0$), and consequently $b = 0$. Since $f(1) = 1$ we must have $a = 1$, so $f(z) = z$.