MAT342 Homework 6
Due Friday, March 15

1. Let $\beta(t) = 2 + e^{it}$ for $-\pi \le t \le 0$, and evaluate $\displaystyle\int_\beta (z-2)^3\, dz$.

2. Let γ be the boundary of the rectangle with vertices at the points 1, $1+2i$, $-1+2i$, and -1, oriented in the counterclockwise direction around the origin. Evaluate $\displaystyle\int_\gamma e^{i\bar{z}}\, dz$.

3. Let γ be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ lying in the first quadrant. Without evaluating the integral, show that

$$\left| \int_\gamma \frac{z+4}{z^3-1}\, dz \right| \le \frac{6\pi}{7}.$$

4. (a) Let $f_1(z)$ be the branch of $z^{1/2}$ given by

$$f_1(re^{i\theta}) = \sqrt{r}\, e^{i\theta/2} \qquad \text{with } r > 0, \quad -\frac{\pi}{2} < \theta < \frac{3\pi}{2},$$

and let γ be any contour lying in the *upper* half-plane (that is, with $\operatorname{Im}\gamma(t) > 0$ except at the endpoints of γ) which goes from 4 to -4. Use an antiderivative of f_1 to compute $\displaystyle\int_\gamma z^{1/2}\, dz$.

(b) Now let $f_2(z)$ be the branch of $z^{1/2}$ given by

$$f_2(re^{i\theta}) = \sqrt{r}\, e^{i\theta/2} \qquad \text{with } r > 0, \quad \frac{\pi}{2} < \theta < \frac{5\pi}{2},$$

and let β be any contour lying in the *lower* half-plane which goes from 4 to -4. Compute $\displaystyle\int_\beta z^{1/2}\, dz$ using an antiderivative of f_2.

(c) Observe that $f_1(z) = f_2(z)$ for z in a neighborhood of -4. Use the results of parts (a) and (b) to calculate

$$\int_{\mathcal{C}} z^{1/2}\, dz$$

where $\mathcal{C} = \gamma - \beta$ is a positively oriented closed countour around the origin.

5. Let $\mathcal{C}$ be the positively oriented circle of radius $R > 0$ centered at z_0 and parameterized as $z = z_0 + Re^{i\theta}$ for $-\pi \le \theta \le \pi$. Show that

$$\int_{\mathcal{C}} (z-z_0)^{n-1}\, dz = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2, \pm 3, \ldots \\ 2\pi i & \text{when } n = 0. \end{cases}$$