MAT342 Homework 6 Solutions

Due Friday, March 15

1. Let $\beta(t) = 2 + e^{it}$ for $-\pi \le t \le 0$, and evaluate $\int_\beta (z-2)^3\,dz$.

Since $(z-2)^3$ is entire, we can integrate along any path from 1 to 3, for example, along the real line. So we have

$$\int_\beta (z-2)^3\,dz = \int_1^3 (z-2)^3\,dz = \tfrac{1}{4}(z-2)^4\Big|_1^3 = \tfrac{1}{4}(-1)^4 - \tfrac{1}{4} = 0\ .$$

2. Let γ be the boundary of the rectangle with vertices at the points 1, $1+2i$, $-1+2i$, and -1, oriented in the counterclockwise direction around the origin. Evaluate $\int_\gamma e^{i\bar{z}}\,dz$.

Since $e^{i\bar{z}}$ is not analytic, let's parameterize the curves and calculate the integral directly. Let the four sides be $R(t) = 1 + 2ti$, $T(t) = 1 - 2t + 2i$, $L(t) = -1 + 2i - 2ti$, and $B(t) = -1 + 2t$ with $0 \le t \le 1$. Then we have

$$\begin{aligned}
\int_R e^{i\bar{z}}\,dz &= \int_0^1 e^{i(1-2ti)} \cdot 2i\,dt &&= i\,e^{i+2t}\Big|_0^1 &&= i(e^{2+i} - e^i)\ ,\\
\int_T e^{i\bar{z}}\,dz &= \int_0^1 e^{i(1-2t-2i)} \cdot (-2)\,dt &&= -i\,e^{2+i(1-2t)}\Big|_0^1 &&= -i(e^{2-i} - e^{2+i})\ ,\\
\int_L e^{i\bar{z}}\,dz &= \int_0^1 e^{i(-1+2i-2ti)} \cdot (-2i)\,dt &&= -i\,e^{i+2t}\Big|_0^1 &&= -i(e^{2+i} - e^i)\ ,\\
\int_B e^{i\bar{z}}\,dz &= \int_0^1 e^{i(-1+2t)} \cdot 2\,dt &&= i\,e^{i(1-2t)}\Big|_0^1 &&= i(e^{-i} - e^i)\ ,
\end{aligned}$$

and so

$$\begin{aligned}
\int_\gamma e^{i\bar{z}}\,dz &= i(e^{2+i} - e^i - e^{2-i} + e^{2+i} - e^{2+i} + e^i + e^{-i} - e^i) = i\Big((e^{-i} - e^i) + (e^{2+i} - e^{2-i})\Big)\\
&= i(e^2 - 1)(e^i - e^{-i}) = -2(e^2-1)\sin 1 \approx -10.75241\ .
\end{aligned}$$

(Unless I screwed up, which happens.)

3. Let γ be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ lying in the first quadrant. Without evaluating the integral, show that

$$\left|\int_\gamma \frac{z+4}{z^3-1}\,dz\right| \le \frac{6\pi}{7}\ .$$

Here we use the fact that if γ is a contour of length L and $|f(z)| < M$ along γ, $|\int_\gamma f(z)\,dz| < ML$. Since γ is a quarter of a circle of radius 2, it is of length $L = \pi$.
For $|z| = 2$, we have $|z+4| \ge 6$ and $|z^3 - 1| \le 7$, so $|f(z)| \le 6/7$ along γ . This gives the result.

4. (a) Let $f_1(z)$ be the branch of $z^{1/2}$ given by

$$f_1(re^{i\theta}) = \sqrt{r}\,e^{i\theta/2} \qquad \text{with } r > 0, \quad -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\,,$$

and let γ be any contour lying in the *upper* half-plane (that is, with $\operatorname{Im}\gamma(t) > 0$ except at the endpoints of γ) which goes from 4 to -4. Use an antiderivative of f_1 to compute $\int_\gamma z^{1/2}\,dz$.

Let $F_1(z) = \frac{2}{3}z^{3/2}$, with $z \neq 0$ and $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$. This is analytic on its domain, and $F'(z) = z^{1/2} = f_1(z)$. Observe that $F_1(4) = 16/3$ and $F_1(-4) = -16i/3$, so

$$\int_\gamma f_1(z)\,dz = F_1(z)\Big|_4^{-4} = -\frac{16(1+i)}{3}\,.$$

(b) Now let $f_2(z)$ be the branch of $z^{1/2}$ given by

$$f_2(re^{i\theta}) = \sqrt{r}\,e^{i\theta/2} \qquad \text{with } r > 0, \quad \frac{\pi}{2} < \theta < \frac{5\pi}{2}\,,$$

and let β be any contour lying in the *lower* half-plane which goes from 4 to -4. Compute $\int_\beta z^{1/2}\,dz$ using an antiderivative of f_2.

Let $F_2(z) = \frac{2}{3}z^{3/2}$ with the same domain as f_2; then F_2 is an antiderivative of f_2 on its domain. Here we again have $F_2(-4) = -16i/3$, but in this branch $4 = 2e^{2\pi i}$, so $F_2(4) = \frac{2}{3}\cdot 8e^{3\pi i} = -16/3$. This means

$$\int_\beta f_2(z)\,dz = F_2(-4) - F_2(4) = -16i/3 + 16/3 = \frac{16(1-i)}{3}\,.$$

(c) Observe that $f_1(z) = f_2(z)$ for z in a neighborhood of -4. Use the results of parts (a) and (b) to calculate

$$\int_{\mathcal{C}} z^{1/2}\,dz$$

where $\mathcal{C} = \gamma - \beta$ is a positively oriented closed countour around the origin.

As noted, $F_1(-4) = -16i/3 = F_2(-4)$, and so if we take $\mathcal{C} = \gamma - \beta$, we will have

$$\begin{aligned}\int_{\mathcal{C}} z^{1/2}\,dz &= \int_\gamma z^{1/2}\,dz + \int_{-\beta} z^{1/2}\,dz = \int_\gamma z^{1/2}\,dz - \int_\beta z^{1/2}\,dz \\ &= \big(F_1(-4) - F_1(4)\big) - \big(F_2(-4) - F_2(4)\big) = F_2(4) - F_1(4) = -16/3 - 16/3 = -32/3\,.\end{aligned}$$

5. Let $\mathcal{C}$ be the positively oriented circle of radius $R > 0$ centered at z_0 and parameterized as $z = z_0 + Re^{i\theta}$ for $-\pi \leq \theta \leq \pi$. Show that

$$\int_{\mathcal{C}} (z - z_0)^{n-1}\, dz = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2, \pm 3, \dots \\ 2\pi i & \text{when } n = 0. \end{cases}$$

First, consider n as any nonzero integer, that is, $n \in \{\pm 1, \pm 2, \pm 3, \dots\}$. In this case, $(z - z_0)^{n-1}$ has an antiderivative $\frac{1}{n}(z - z_0)^n$ which is analytic on a neighborhood of $\mathcal{C}$. (In fact, this antiderivative is entire if n is a positive integer and analytic $\mathbb{C} - \{z_0\}$ when n is a negative integer.) If a function has an analytic antiderivative defined on a neighborhood of a closed contour $\mathcal{C}$, then its integral over that contour is zero.

Now consider $n = 0$. In this case, there is an antiderivative which is $F(z) = \log(z - z_0)$, but this function is only analytic on a domain which omits a simple curve joining z_0 to ∞. That is, we have to choose an appropiate branch of the logarithm. Given the parameterization of $\mathcal{C}$, it makes sense to choose the principal branch $\operatorname{Log}(z - z_0)$ (but in fact, we can choose any branch).

Then observe that

$$\lim_{\theta \to \pi^-} F(z_0 + Re^{i\theta}) = \ln R + \pi i \qquad \text{but} \qquad \lim_{\theta \to \pi^+} F(z_0 + Re^{i\theta}) = \ln R - \pi i$$

and hence $\displaystyle\int_{\mathcal{C}} \frac{dz}{z - z_0} = 2\pi i$.

If you choose a different branch of the logarithm, a little more work is needed but the result is the same.