MAT342 Homework 5

Due Wednesday, March 6

1. We (supposedly, but see quiz 2) know that if the derivative of a function f exists at z_0 then the Cauchy-Riemann equations must hold, but that the converse is not necessarily true (additional conditions are needed, such as continuity of partials).

Show that for the function

$$f(z) = f(x+iy) = \begin{cases} \frac{x^2-y^2-2xy\,i}{x+iy} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

the Cauchy-Riemann equations hold at $z = 0$ but f is not differentiable at $z = 0$.

Hint: consider $z \to 0$ along the real axis and along the line $y = x$.

2. Explain why $\mathrm{Re}(e^{1/z^2})$ is harmonic everywhere except at the origin.

3. (a) Assume that $w \in \mathbb{C}$ with $\alpha < \mathrm{Im}\, w < \alpha + 2\pi$ for some (fixed) $\alpha \in \mathbb{R}$. Show that for $z = re^{i\theta}$, when the branch of logarithm

$$\log z = \ln r + i\theta, \quad \text{with } r > 0, \alpha < \theta < \alpha + 2\pi$$

is used, we always have $\log(e^w) = w$.

(b) Give an branch of the logarithm that ensures that for $\beta = 1 + i$ we have

$$\log(\beta^8) = 8\log(\beta).$$

(c) For the same β as in the previous part, give a branch of the logarithm for which

$$\log(\beta^8) \neq 8\log(\beta).$$

4. Calculate each of the following. Keep in mind that these expressions can be *multivalued*.

(a) $(-1+i\sqrt{3})^{3/2}$ (b) i^π (c) π^i (d) i^{-2i}

5. Find all roots of the equation $\sin z = \cosh 4$ by equating the real parts of both sides, then equating the imaginary parts.

6. Show that $\sinh z = 0$ if and only if $z = in\pi$ with $n \in \mathbb{Z}$. You may use facts we already established about e^z, $\sin z$ and $\cos z$ without reproving them explicitly.

7. Evaluate the integrals below.

(a) $\displaystyle\int_0^1 (1+it)^2\, dt$ (b) $\displaystyle\int_0^{\pi/2} e^{2ti}\, dt$