MAT342 Homework 5 Solutions
Due Wednesday, March 6

1. We (supposedly, but see quiz 2) know that if the derivative of a function f exists at z_0 then the Cauchy-Riemann equations must hold, but that the converse is not necessarily true (additional conditions are needed, such as continuity of partials).

Show that for the function

$$f(z) = f(x+iy) = \begin{cases} \frac{x^2-y^2-2xy\,i}{x+iy} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

the Cauchy-Riemann equations hold at $z=0$ but f is not differentiable at $z=0$.
Hint: consider $z \to 0$ along the real axis and along the line $y=x$.

To check the Cauchy-Riemann equations, we must check that $u_x = v_y$ and $u_y = -v_x$, where $f(x+iy) = u(x,y)+iv(x,y)$. Observe that for $z \neq 0$, $f(z) = \frac{\bar{z}^2}{z} = \frac{\bar{z}^3}{|z|}$ (this isn't strictly necessary, but is easier for me to think of.) So for $z = x+iy$, we have

$$f(z) = \frac{(x-iy)^2}{x+iy} = \frac{(x-iy)^3}{x^2+y^2} = \frac{x^3-3xy^2}{x^2+y^2} + i\frac{y^3-3x^2y}{x^2+y^2}.$$

Now we calculate

$$u_x(0,0) = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0} \frac{x^3/x^2}{x} = \lim_{x\to 0} 1 = 1.$$

Similarly,

$$v_y(0,0) = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0} \frac{y^3/y^2}{y} = 1,$$

so $u_x(0,0) = 1 = v_y(0,0)$. The other two partials are even easier:

$$u_y = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0} \frac{0}{y^3} = 0 \quad \text{and} \quad v_x = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \lim_{y\to 0} \frac{0}{x^3} = 0$$

and hence $u_y(0,0) = 0 = -v_x(0,0)$, showing the Cauchy-Riemann equations hold.

To see that $f(z)$ is not differentiable at $z=0$, first look at the derivative (as a limit) along the real axis (which is actually just $u_x + iv_x$), where we have

$$\lim_{x\to 0} \frac{f(x+0i)-f(0)}{x} = \lim_{x\to 0} \frac{x^2/x}{x} = 1.$$

But if we look along the line $x=y$ (that is, $\operatorname{Re} z = \operatorname{Im} z$), we get

$$\lim_{x\to 0} \frac{f(x+ix)-f(0)}{x+ix} = \lim_{x\to 0} \frac{x^2-x^2-2x^2 i}{(x+ix)\cdot x} \lim_{x\to 0} \frac{-2i}{1+i} = -1-i.$$

Obviously, these are not the same. The complex derivative of f does not exist at $z=0$ (since we get different values of the limit by different approaches to 0), despite the Cauchy-Riemann equations being satisfied there.

2. Explain why $\operatorname{Re}(e^{1/z^2})$ is harmonic everywhere except at the origin.

On any domain not containing the origin, $1/z^2$ is analytic. Since the exponential function is entire, the composition e^{1/z^2} is analytic on any domain avoiding the origin.

The real part of any analytic function is always a harmonic function.

If you are a masochist, you can write the function out in real and imaginary parts, and then compute that $u_{xx} + v_{yy} = 0$. But that is a lot of work, and I certainly don't want to do that.

3. (a) Assume that $w \in \mathbb{C}$ with $\alpha < \operatorname{Im} w < \alpha + 2\pi$ for some (fixed) $\alpha \in \mathbb{R}$. Show that for $z = re^{i\theta}$, when the branch of logarithm

$$\log z = \ln r + i\theta, \quad \text{with } r > 0, \alpha < \theta < \alpha + 2\pi$$

is used, we always have $\log(e^w) = w$.

Let's write $w = x + iy$, where $x \in \mathbb{R}$ and $\alpha < y < \alpha + 2\pi$. Then $e^w = e^{x+iy} = e^x e^{iy}$, and

$$\ell og(e^w) = \{\ln(e^x) + (y + 2\pi n)i\}, \quad n \in \mathbb{Z} \;,$$

where ℓog represents the multivalued logarithm and (as usual) ln represents the logarithm from $\mathbb{R}^+$ to $\mathbb{R}$. But the branch of the logarithm taken in this problem corresponds to $n = 0$ (since $\alpha < y < \alpha + 2\pi$), and we have

$$\log(e^w) = \ln(e^x) + iy = x + iy = w \;.$$

(b) Give an branch of the logarithm that ensures that for $\beta = 1 + i$ we have

$$\log(\beta^8) = 8\log(\beta).$$

Observe that $\beta = 1 + i = \sqrt{2}e^{i\pi/4}$ and $\beta^8 = (\sqrt{2}e^{i\pi/4})^8 = 16$. Consequently, if the branch of the logarithm chosen has $\log\beta = \frac{\ln 2}{2} + i\frac{\pi}{4}$, we also need $\log\beta^8 = \log(16) = 4\ln 2 + 2\pi i$. This means we need to take a branch cut of argument α where $0 < \alpha < \pi/4$. For example, we may choose $\alpha = \pi/8$. Then

$$\log{(1+i)}^8 = \log(16) = 4\ln 2 + 2\pi i = 8(\tfrac{\ln 2}{2} + \tfrac{\pi}{4}i) = 8\log(1+i) \;,$$

where log is the branch of the logarithm with $\log(z) = \ln|z| + i\arg z, \quad \pi/8 < \arg z < 17\pi/8.$

(c) For the same β as in the previous part, give a branch of the logarithm for which

$$\log(\beta^8) \neq 8\log(\beta).$$

Again, assuming we take a branch so that $\operatorname{Im}\log\beta = \pi/4$, any branch cut of argument α where $\alpha \geq \pi/4$ or $\alpha \leq 0$ will do. For example, for the principal branch of the logarithm (that is, with $\alpha = -\pi$), we have

$$\operatorname{Log}\beta^8 = \operatorname{Log}16 = \ln 16 = 4\ln 2 \quad \neq \quad 8\operatorname{Log}(1+i) = 8(\tfrac{\ln 2}{2} + \tfrac{\pi}{4}i) = 4\ln 2 + 2\pi i \;.$$

4. Calculate each of the following. Keep in mind that these expressions can be *multivalued*.

(a) $(-1 + i\sqrt{3})^{3/2} \quad = \left(2e^{2\pi i/3}\right)^{3/2} = \left(8e^{2\pi i}\right)^{1/2} = (8)^{1/2} = \{2\sqrt{2}, -2\sqrt{2}\}.$

Note that this is the same result that you would get by writing

$$\exp(\tfrac{3}{2}\log(-1+\sqrt{3}i)) = \exp\left(\ln(2^{3/2}) + \tfrac{3}{2}(2\pi/3 + 2n\pi)i\right) = \sqrt{8}\,e^{(\pi+3n\pi)i} \quad \text{for } n \in \mathbb{Z}$$

since for n even, $e^{(3n+1)\pi i} = -1$ and for n odd, $e^{(3n+1)\pi i} = 1$.

(b) $i^\pi \quad = e^{\pi \log i} = \exp\left(\pi(i\pi/2 + 2n\pi i)\right) = \exp\left(i(\pi^2/2 + 2n\pi^2)\right) = \cos\left(\frac{4n+1}{2}\pi^2\right) + i\sin\left(\frac{4n+1}{2}\pi^2\right)$ for $n \in \mathbb{Z}$.

(c) $\pi^i \quad = e^{i\log\pi} = e^{i(\ln\pi + 2n\pi i)} == e^{i\ln\pi - 2n\pi)} = e^{2n\pi}(\cos(\ln(\pi)) + i\sin(\ln(\pi)))$ for $n \in \mathbb{Z}$.

(d) $i^{-2i} \quad i^{-2i} = \exp(-2i\log i) = \exp(-2i(i\frac{\pi}{2} + 2n\pi i) = e^{(4n+1)\pi}$ for $n \in \mathbb{Z}$.

Observe that the answer to (a) has two values, as you should expect from a square root. By contrast, (b) has infinitely many values distributed densely around the unit circle, the answer to (c) is an infinite set of values along a ray of argument $\ln\pi \approx 1.1447$ and part (d) has infinitely many values, but they are all real.

5. Find all roots of the equation $\sin z = \cosh 4$ by equating the real parts of both sides, then equating the imaginary parts.

Recall that $\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$, and writing $z = x + iy$ gives the equation

$$e^{ix-y} - e^{-ix+y} = 2i\cosh(4)$$

which we can rewrite as

$$e^{-y}(\cos x + i\sin x) - e^{y}\left(\cos(-x) + i\sin(-x)\right) = 2i\cosh(4) \ .$$

Note that $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$. Then equating real parts of both sides, and then imaginary parts yields the two equations

$$\cos x\,(e^{-y} - e^{y}) = 0 \qquad\qquad \sin x\,(e^{-y} + e^{y}) = 2\cosh(4) \ .$$

The equation on the left tells us that either $y = 0$ or $x = \frac{\pi}{2} + 2n\pi$ for some $n \in \mathbb{Z}$. But from the right-hand equation, we cannot have $y = 0$ (since $\cosh(4) = \frac{1}{2}(e^4 + e^{-4}) \neq 0$). Hence $\sin x = 1$.

Using this, we can rewrite the right-hand equation as $e^{-y} + e^{y} = e^4 + e^{-4}$. Consequently, $y = \pm 4$. There can be no other solutions, since $e^{-y} + e^{y}$ is monotonically decreasing for $y < 0$ and increasing for $y > 0$.

Hence, any solution to the equation is of the form

$$z = \frac{\pi}{2} + 2n\pi \pm 4i, \quad \text{for } n \in \mathbb{Z}$$

(and every such number is a solution).

6. Show that $\sinh z = 0$ if and only if $z = in\pi$ with $n \in \mathbb{Z}$. You may use facts we already established about e^z, $\sin z$ and $\cos z$ without reproving them explicitly.

Recall that $\sinh z = -i\sin(iz)$. We already know that all the zeros of $\sin z$ are real numbers of the form $n\pi$ with $n \in \mathbb{Z}$, and hence the zeros of $\sinh z$ are exactly the points on the imaginary axis of the form $n\pi i$ for $n \in \mathbb{Z}$.

7. Evaluate the integrals below.

(a) $\displaystyle\int_0^1 (1+it)^2\,dt \quad = \int_0^1 1 - t^2\,dt + i\int_0^1 2t\,dt = t - t^3/3\Big|_0^1 + it^2\Big|_0^1 = \frac{2}{3} + i$.

(b) $\displaystyle\int_0^{\pi/2} e^{2ti}\,dt \quad = \int_0^{\pi/2} \cos(2t)\,dt + i\int_0^{\pi/2} \sin(2t)\,dt = \frac{1}{2}\sin(2t)\Big|_0^{\pi/2} - \frac{i}{2}\cos(2t)\Big|_0^{\pi/2} = -i$.