MAT342 Homework 4

Due Wednesday, February 27

1. Section 2.24 of the text discusses the Cauchy-Riemann equations when z is interpreted in polar form. That is, writing $z = re^{i\theta}$ with $r > 0$, if $f(z) = u(r,\theta) + iv(r,\theta)$ is defined in a neighborhood $\mathcal{U}$ of z_0 and the partials of u and v with respect to r and θ are defined for $z \in \mathcal{U}$ and are continuous at z_0, then $f'(z_0)$ exists whenever

$$ru_r = v_\theta \qquad \text{and} \qquad u_\theta = -rv_r \ .$$

Show that for $z \in \mathcal{U}$, we have $f'(z) = e^{-i\theta}(u_r + iv_r)$ when it exists.

2. (a) Show that if $f = u + iv$ is analytic in a domain $\mathcal{D}$, then $(\nabla u) \cdot (\nabla v) = 0$, where ∇ is the gradient operator from multivariable calculus.

(b) Consider the level curves given by $u(x,y) = a$ and $v(x,y) = b$. What does the previous part tell you about how these curves intersect at a point $f(z_0) = a + ib$?

3. (a) Show that the function $h(x,y) = x^3 - 3xy^2 + 2x$ is harmonic.

(b) Find an analytic function $f(z) = f(x+iy)$ so that $f(z) = h(x,y) = \text{Re}(f(z))$. That is, find $v(x,y)$ so that $x^3 - 3xy^2 + 2x + iv(x,y)$ is analytic. The function $v(x,y)$ is called the **harmonic conjugate** of h.
(Hint: Since $f(x,y) = h(x,y) + iv(x,y)$ is analytic, you can determine what v_x and v_y must be. Then integrating with v_y with respect to y then gives you an expression that differs from $v(x,y)$ by a function depending on x alone. Similarly, you can integrate v_x with respect to x to get $v(x,y) + k(y)$; equating the two gives you $v(x,y)$.) Or maybe you can just guess the answer, and check that it works.)

4. Write $z = re^{i\theta}$ and consider the function $f_1(z) = \sqrt{r}e^{i\theta/2}$ for $r > 0$ and $0 < \theta < \pi$, which is analytic on the upper half-plane.

(a) Now let $f_2(z) = \sqrt{r}e^{i\theta/2}$ with $r > 0$ and $-3\pi/4 < \theta < \pi/4$. Discuss why f_2 is an analytic continuation of f_1.

(b) Similarly, let $f_3(z) = \sqrt{r}e^{i\theta/2}$ with $r > 0$ and $3\pi/4 < \theta < 7\pi/4$. Is f_3 an analytic continuation of f_1?

(c) Compute $f_2(-i)$ and $f_3(-i)$. Do these functions agree? If not, explain.

5. (a) Find all values of $z \in \mathbb{C}$ such that $e^{4z} = 1$.

(b) Find all values of $z \in \mathbb{C}$ such that $e^{iz} = 3$.

6. Recall that Log denotes the principal branch of the logarithm.

(a) Show that $\text{Log}(1+i)^2 = 2\,\text{Log}(1+i)$.

(b) Show that $\text{Log}(-1+i)^2 \neq 2\,\text{Log}(-1+i)$.