Solution to HW 4.

1. Proof: $f'(z) = \lim_{\Delta r \to 0} \dfrac{f(z+\Delta r e^{i\theta}) - f(z)}{\Delta r e^{i\theta}}$

$$= \lim_{\Delta r \to 0} \frac{\left(u(r+\Delta r, \theta) + i\, v(r+\Delta r, \theta)\right) - \left(u(r, \theta) + i\, v(r, \theta)\right)}{\Delta r\, e^{i\theta}}$$

$$= e^{-i\theta}\left[\lim_{\Delta r \to 0} \frac{u(r+\Delta r, \theta) - u(r,\theta)}{\Delta r} + i \lim_{\Delta r \to 0} \frac{v(r+\Delta r, \theta) - v(r,\theta)}{\Delta r}\right]$$

$$= e^{-i\theta}(u_r + i v_r) \qquad \square$$

2. Proof. a) $\nabla u \cdot \nabla v = u_x v_x + u_y v_y \overset{\text{C-R}}{\underset{\text{eqn.}}{=}} u_x v_x + v_x \cdot (-u_x) = 0.$

b) They intersect orthogonally, i.e. at a right angle $\frac{\pi}{2}$. $\square$

3. Proof: a) $\Delta h = h_{xx} + h_{yy}$

$$= 6x - 6x = 0.$$

b) $h_x = \cancel{6x}\ 3x^2 - 3y^2 + 2.$

$h_y = -6xy$. So if v is the harmonic conjugate to h,

we have: $\begin{cases} v_x = +6xy \\ v_y = 3x^2 - 3y^2 + 2. \end{cases}$

Hence $\int_0^{z_0} v_x + i v_y \, dz$ would suffice.

$$= \int_0^{x_0} v_x + i v_y \, dz + \int_{x_0}^{x_0 + i y_0} v_x + i v_y \, dz.$$

$$= \int_0^{x_0} i(3x^2+2)\, dx + i \int_0^{y_0} 6x_0 y + i(3x_0^2 - 3y^2 + 2)\, dy$$

$$= i(x_0^3 + 2x_0) + i\left[3x_0 y_0^2 + i(3x_0^2 y_0 - y_0^3 + 2y_0)\right]$$

$$v(x_0, y_0) = \left(y_0^3 - 2y_0 - 3x_0^2 y_0\right) + i\left(x_0^3 + 3x_0 y_0^2 + 2x_0\right).$$

So if we take $v(0,0)=0$, then

$$v(x_0,y_0)= v(x_0,y_0)-v(x_0,0)+v(x_0,0)-v(0,0)$$

$$= \int_0^{y_0} v_y(x_0,y)\,dy + \int_0^{x_0} v_x(x,0)\,dx$$

$$= \int_0^{y_0} (3x_0^2-3y^2+2)\,dy + \int_0^{x_0} 0\,dx$$

$$= 3x_0^2 y_0 - y_0^3 + 2y_0 \qquad \square.$$

Pf:

4. a) On the common domain of f_1 and f_2, that is, the region

$$\{ re^{i\theta} \mid r>0,\ 0<\theta<\pi/4 \},$$

both f_1 and f_2 have the same definition, that is $\sqrt{r}\,e^{i\theta/2}$.

Thus f_2 is an analytic continuation of f_1

b). The same argument as a)

c) $f_2(-i) = f_2(1\cdot e^{-i\frac{\pi}{2}}) = \sqrt{1}\, e^{i(-\pi/4)} = e^{-\pi/4}$.

$f_3(-i) = f_3(1\cdot e^{i\cdot\frac{3}{2}\pi}) = \sqrt{1}\,e^{i3\pi/4} = e^{3\pi/4}$. $\quad f_2(-i)\neq f_3(-i)$ $\quad\square$.

5. a) $e^{4z}=1 \iff 4z=2k\pi i \iff z=\frac{k\pi i}{2} \quad k\in\mathbb{Z}$

b) $e^{iz}=3 \iff iz=\ln 3+2k\pi i \iff z=2k\pi - i\ln 3 \quad k\in\mathbb{Z}$

6. Recall that for the principal branch of Log, we choose the branch cut where $-\pi<\arg<\pi$.

a) $(1+i)=\sqrt{2}e^{i\pi/4}$, $\quad \log((1+i)^2) = \log(2e^{i\pi/2}) = \log 2 + i\pi/2$. $\quad 2\log(1+i) = 2(\log(\sqrt{2})+i\frac{\pi}{4}) = \log 2 + i\pi/2$.

b) $(-1+i) = \sqrt{2}e^{i3\pi/4}$ $\quad 2\log(-1+i) = 2(\ln\sqrt{2} + i3\pi/4) = \ln 2 + i\,3\pi/2$.

$(-1+i)^2 = -2i = 2e^{(-\pi/2)} \Rightarrow \log(-1+i)^2 = \ln 2 - i\pi/2$ $\quad\square$.