MAT342 Homework 3
Due Wednesday, February 20

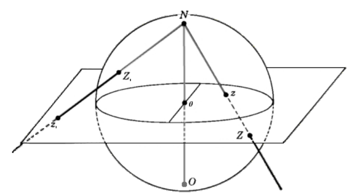
1. For fixed complex numbers a, b, c, and d, let $T(z) = \dfrac{az+b}{cz+d}$ with $ad - bc \neq 0$. The map T is called a **Möbius Transformation** or a **linear fractional transformation**; these mappings play an important role in many areas of mathematics, especially in complex analysis and non-Euclidean geometry, and are closely connected with Einstein's Theory of Relativity.

(a) If $c = 0$, compute $\lim_{z\to\infty} T(z)$.

(b) Assuming $c \neq 0$, compute $\lim_{z\to\infty} T(z)$.

(c) Again assuming $c \neq 0$, compute $\lim_{z\to -d/c} T(z)$.

2. Recall the usual stereographic projection of $\mathbb{C}$ to the Riemann sphere $\overline{\mathbb{C}}$, where a point z in the plane corresponds to a point Z on the sphere when the line (in $\mathbb{R}^3$) joining the north pole N to z intersects the sphere at Z. Now consider the (inverse) stereographic projection taking a point Z on the sphere back to some w in the plane by reversing the process, but instead using the line joining Z with the *south pole* (labeled O in the figure), giving w as the intersection of this line with the plane. The composition of these two gives rise to a map $f\colon z \mapsto w$ of the plane $\mathbb{C}$ to itself. What is this mapping? Give a formula for w in terms of z and familiar functions.



3. Prove that the function $f(z) = z \cdot \mathrm{Im}(z)$ is differentiable only at $z = 0$ and is not differentiable at any nonzero $z \in \mathbb{C}$. What is $f'(0)$?

4. Let $z = x + iy$. Show that the function

$$f(z) = e^{x^2-y^2}(\cos(2xy) + i\sin(2xy))$$

is entire, and find $f'(z)$ for $z \in \mathbb{C}$.

5. For each of the functions listed below, determine at what points $z \in \mathbb{C}$ is not differentiable (and where it is). When the function is differentiable at z, calculate its derivative. Justify your answers fully.

(a) $f(x+iy) = e^{-x}e^{iy}$

(b) $g(x+iy) = e^{-x}e^{-iy}$

(c) $h(z) = z - \overline{z}$

(d) $k(z) = 1/z^2$

6. Let $f(z) = (\overline{z})^2 - 1$. Show that $f(z)$ is not analytic on any domain in $\mathbb{C}$, but the function $g(z) = f(f(z))$ is entire.