MAT342 Homework 3

Due Wednesday, February 20

1. For fixed complex numbers a, b, c, and d, let $T(z) = \dfrac{az+b}{cz+d}$ with $ad - bc \neq 0$. The map T is called a **Möbius Transformation** or a **linear fractional transformation**; these mappings play an important role in many areas of mathematics, especially in complex analysis and non-Euclidean geometry, and are closely connected with Einstein's Theory of Relativity.

(a) If $c = 0$, compute $\lim_{z\to\infty} T(z)$.

We have $\lim_{z\to\infty} \dfrac{az+b}{d} = \infty$, since, letting $w = 1/z$, we have

$$\lim_{w\to 0} \frac{1}{T(1/w)} = \lim_{w\to 0} \frac{d}{a/w+b} = \lim_{w\to 0} \frac{dw}{a+bw} = 0.$$

(b) Assuming $c \neq 0$, compute $\lim_{z\to\infty} T(z)$.

We have $\lim_{z\to\infty} \dfrac{az+b}{cz+d} = \dfrac{a}{c}$, since

$$\lim_{w\to 0} \frac{1}{T(1/w)} = \lim_{w\to 0} \frac{c/w+d}{a/w+b} = \lim_{w\to 0} \frac{c+dw}{a+bw} = \frac{c}{a}.$$

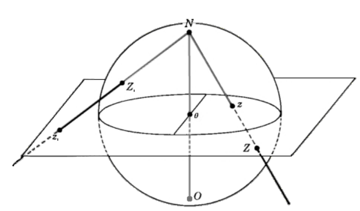
Hence the original limit is $\dfrac{1}{c/a} = \dfrac{a}{c}$.

(c) Again assuming $c \neq 0$, compute $\lim_{z\to -d/c} T(z)$.

To see that $\lim_{z\to -d/c} T(z) = \infty$, we show that the limit of $1/T(z) = 0$. Note that $1/T(z)$ is continuous near $z = -d/c$, since it is a rational function and $ad - bc \neq 0$.

$$\lim_{z\to -d/c} \frac{1}{T(z)} = \lim_{z\to -d/c} \frac{cz+d}{az+b} = \frac{c(-d/c)+d}{a(-d/c)+b} = \frac{-d+d}{b-ac/d} = 0.$$

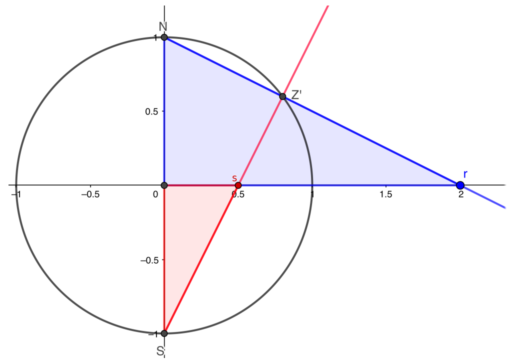
2. Recall the usual stereographic projection of $\mathbb{C}$ to the Riemann sphere $\overline{\mathbb{C}}$, where a point z in the plane corresponds to a point Z on the sphere when the line (in $\mathbb{R}^3$) joining the north pole N to z intersects the sphere at Z. Now consider the (inverse) stereographic projection taking a point Z on the sphere back to some w in the plane by reversing the process, but instead using the line joining Z with the *south pole* (labeled O in the figure), giving w as the intersection of this line with the plane. The composition of these two gives rise to a map $f\colon z \mapsto w$ of the plane $\mathbb{C}$ to itself. What is this mapping? Give a formula for w in terms of z and familiar functions.



The easiest way to do this (for me) is first notice that the problem can be reduced to one in Euclidean geometry (ie, high-school geometry).

If we write $z = a+ib$ as $(a,b,0) \in \mathbb{R}^3$, the north pole as $N = (0,0,1) \in \mathbb{R}^3$, and the south pole as $S = (0,0,-1)$, then all of these points lie in the same plane (namely, the one perpendicular to $\mathbb{C}$ containing the $N-S$ axis). This means, of course, that the resulting points Z and w also lie in this plane. In particular, this means the points $z \in \mathbb{C}$ and $w \in \mathbb{C}$ have the same argument since they lie on the same ray through the origin in $\mathbb{C}$.

Observe that if $z = 0$, the image of the first projection to the sphere is the south pole, and the second projection back to the complex plane is not well defined (the only thing that makes sense is to call it ∞, but this is not a complex number). So we may assume $z \neq 0$.



Writing $z = re^{i\theta}$, w must be of the form $w = se^{i\theta}$ since $\arg z = \arg w$. We now have to write s in terms of r. Looking at the image to the right, there are two relevant right triangles: the blue one (with vertices N, r, and 0) and the red one (with vertices S, s, and 0). (A similar picture occurs if $r < 1$, but with s outside the circle; if $r = 1$ then $s = 1$.) These two triangles are similar, and so the ratio of corresponding sides must be preserved. In particular, we must have the ratio of the short leg to the long leg equal. That is, $\frac{1}{r} = \frac{s}{1}$.
Consequently, we have

$$f(z) = f(re^{i\theta}) = se^{i\theta} = \tfrac{1}{r}e^{i\theta} = \frac{1}{re^{-i\theta}} = \frac{1}{\bar{z}} .$$

If you wanted to, you could do everthing in coordinates in $\mathbb{R}^3$. Again, write $z = (a,b,0)$ and $N = (0,0,1)$. Then the line through z and N can be written as $\{x = at, y = bt, z = 1-t\}$, which intersects the sphere $x^2 + y^2 + z^2 = 1$ at $Z = \left(\frac{2a}{a^2+b^2+1}, \frac{2b}{a^2+b^2+1}, 1 - \frac{2}{a^2+b^2+1}\right)$. Then you can do a similar thing to figure out the coordinates of w. This is not my favorite way, by any means. But tastes differ, and this way isn't wrong. There are also other correct variations of ways to do this problem.

3. Prove that the function $f(z) = z \cdot \operatorname{Im}(z)$ is differentiable only at $z = 0$ and is not differentiable at any nonzero $z \in \mathbb{C}$. What is $f'(0)$?

Write $f(x+iy) = y(x+iy) = xy + iy^2 = u + iv$. Taking partials gives

$$u_x = y \quad u_y = x \quad v_x = 0 \quad v_y = 2y.$$

For the Cauchy-Riemann equations to hold, we have to have $x = 0$ and $y = 2y$, so $y = 0$. This means the function is only differentiable at $z = 0$, and at this point, $f'(0) = 0$.

4. Let $z = x + iy$. Show that the function

$$f(z) = e^{x^2-y^2}(\cos(2xy) + i\sin(2xy))$$

is entire, and find $f'(z)$ for $z \in \mathbb{C}$.

Writing $z = x + iy$, observe that $z^2 = (x^2 - y^2) + i(2xy)$. Consequently, by Euler's formula

$$e^{z^2} = e^{(x^2-y^2)+2xyi} = e^{x^2-y^2}(\cos(2xy) + i\sin(2xy)) = f(z).$$

Since f is the composition of two entire functions, it is entire. Furthermore, the ordinary rules of differentiation apply, so using the chain rule, we have

$$f'(z) = 2ze^{z^2} = 2e^{x^2-y^2}\left((x+iy)\cos(2xy) - (x+iy)\sin(2xy)\right).$$

You can, of course, do everything in terms of x and y, and check the Cauchy-Riemann equations if it makes you happy to do it this way. But it makes me sad, so I won't.

5. For each of the functions listed below, determine at what points $z \in \mathbb{C}$ is not differentiable (and where it is). When the function is differentiable at z, calculate its derivative. Justify your answers fully.

(a) $f(x+iy) = e^{-x}e^{iy}$

Writing in terms of z, we have $f(z) = e^{-x+iy} = e^{-x}(\cos y + i\sin y) = e^{-\bar{z}}$. This is not analytic, but the derivative could exist at isolated points, so checking the Cauchy-Riemann equations gives us

$$u_x = -e^{-x}\cos y \quad u_y = -e^{-x}\sin y \quad v_x = -e^{-x}\sin y \quad v_y = e^{-x}\cos y.$$

We cannot have $u_x = v_y$ and $u_y = -v_x$, since if $-e^{-x}\cos y = e^{-x}\cos y$, we must have $\cos y = 0$. But then $\sin y = 1$, and so $v_x = u_y$. Hence f is nowhere differentiable.

(b) $g(x+iy) = e^{-x}e^{-iy}$

Observe that $g(z) = e^{-z}$, so g is entire. Or you can check the C-R equations.
We have $g'(z) = -e^{-z}$ for any $z \in \mathbb{C}$.

(c) $h(z) = z - \bar{z}$

Note that $h(z) = 2\operatorname{Im} z$, and since $u_x = 0$ and $v_y = 2$, the C-R equations fail everywhere, and the function h is nowhere differentiable.

(d) $k(z) = 1/z^2$

We know from class that $1/z$ is analytic on $\mathbb{C} - \{0\}$, so since $k(z) = \frac{1}{z} \cdot \frac{1}{z}$, we must have $k(z)$ differentiable at all $z \neq 0$. Since k is not even defined for $z = 0$, it is not differentiable there.
Alternatively, you can write $k(x+iy) = \dfrac{1}{x^2 - y^2 + 2xyi} = \dfrac{x^2 + y^2 - 2xyi}{(x^2 - y^2)^2 + 4x^2y^2}$ and check the C-R equations. It's a little messy, and I'd rather not.
For $z \neq 0$, we have $k'(z) = -2/z^3$.

6. Let $f(z) = (\bar{z})^2 - 1$. Show that $f(z)$ is not analytic on any domain in $\mathbb{C}$, but the function $g(z) = f(f(z))$ is entire.

Since $f(x+iy) = (x - iy)^2 - 1 = x^2 - y^2 - 1 - 2xyi$, we can see that

$$u_x = 2x \quad u_y = -2y \quad v_x = -2y \quad v_y = -2x,$$

so the Cauchy-Riemann equations can only hold if $x = 0$ and $y = 0$; that is, $f'(0) = 0$. But to be analytic, we must have the C-R equations satisfied in a neighborhood, not just at a point. So f is not analytic on any domain in $\mathbb{C}$.

On the other hand, observe that

$$g(z) = f(f(z)) = (\overline{\bar{z}^2 - 1})^2 - 1 = (z^2 - 1)^2 - 1 = z^4 - 2z^2.$$

Since $g(z)$ is a polynomial, it is entire.
(If you want to check the C-R equations, have fun. I don't want to.)