MAT342 Homework 2

Due Wednesday, February 13

1. Derive the trigonometric identities

$$\cos(3\theta) = \cos^3\theta - 3\cos\theta\ \sin^2\theta \quad \text{and} \quad \sin(3\theta) = 3\cos^2\theta\ \sin\theta - \sin^3\theta.$$

Hint: use de Moivre's formula.

2. Consider the product $(2+i)(3+i)$, and use this to show that

$$\arctan(1/2) + \arctan(1/3) = \frac{\pi}{4}.$$

3. (a) Show that if a and b are fixed complex numbers with $a \neq b$, then the set

$$\{ z \in \mathbb{C} : |\, z - a| = |z - b| \}$$

describes a line.
(Hint: first think about what kind of set $|z-a| = r$ describes for various values of $r > 0$.)

(b) Show that all nine solutions z_k of the equation $(z-1)^{10} = z^{10}$ must have $\operatorname{Re}(z_k) = \frac{1}{2}$.

4. For each of the sets of complex numbers z below, make a rough sketch of it; then determine whether the set is open, closed, or neither. Also indicate which are domains, and which are bounded.

(a) $\operatorname{Re}(z) > 2$

(b) $|2z+5| \leq 1$

(c) $|z-4| = 6$

(d) $0 \leq \arg(z) \leq \frac{\pi}{4}, (z \neq 0)$

(e) $2 < |z-1| < 3$

(f) $\arg(z) = 1$ or $z = 0$.

5. Determine all the accumulation points of the set

$$\mathcal{W} = \left\{ z \in \mathbb{C} \,\middle|\, 0 \leq \arg(z) \leq \frac{\pi}{4} \text{ and } z \neq 0 \right\}.$$

6. Let ω be any n^{th} root of unity with $\omega \neq 1$. Show that

$$1 + \omega + \omega^2 + \omega^3 + \ldots + \omega^{n-1} = 0.$$

7. Consider the sector

$$\mathcal{V} = \left\{ z \in \mathbb{C} \,\middle|\, 0 \leq \arg(z) \leq \frac{\pi}{4} \text{ and } |z| \leq 1 \right\} \cup \{0\}.$$

Sketch the image of $\mathcal{V}$ under the mappings $s(z) = z^2$, $c(z) = z^3$, and $q(z) = z^4$.

8. For $z \neq 0$, write $z = re^{i\theta}$ with $|z| = r$ and $\operatorname{Arg}(z) = \theta$ (so that $-\pi < \theta \leq \pi$ and $r > 0$). Define $S(z) = \sqrt{r}\, e^{i\theta/2}$ for all $z \neq 0$. Show each of the following:

(a) $\displaystyle\lim_{z \to i} S(z) = \frac{1+i}{\sqrt{2}}$

(b) $\displaystyle\lim_{z \to 0} S(z) = 0$

(c) $\displaystyle\lim_{z \to -1} S(z)$ does not exist.