Solution to HW2, MAT342.

1. **Proof**: Using de Moivre's formula,

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta,$$

and notice that

$$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + i\,3\cos^2\theta\sin\theta$$
$$-3\cos\theta\sin^2\theta - i\sin^3\theta.$$

Compare the real and the imaginery part and we're done. $\square$

2. **Proof**: $\operatorname{Arg}\big((2+i)(3+i)\big) = \operatorname{Arg}(2+i) + \operatorname{Arg}(3+i)$

$$\Rightarrow \underbrace{\operatorname{Arg}(5+5i)}_{=\pi/4} = \arctan\left(\tfrac{1}{2}\right) + \arctan\left(\tfrac{1}{3}\right)$$

$\square$

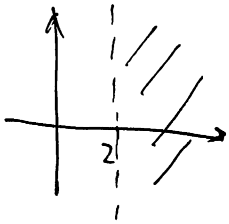
3. **Proof**: a) The set $\{z \in \mathbb{C} \mid |z-a| = |z-b|\}$ consists of points that have the same distance to a and b.

And we know that such property is satisfied if and only if the point lie on the line that is perpendicular to $\overline{ab}$ and going through the midpoint of $\overline{ab}$.

b). Taking the modulus, we have: $|z-1|^{10} = |z|^{10}$. $\Rightarrow |z-1| = |z|$.

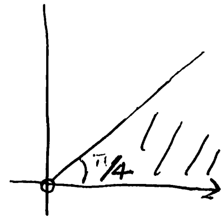
From the discussion in a), we know that $\operatorname{Re}(z_k) = \frac{1}{2}$. $\square$

4. a)



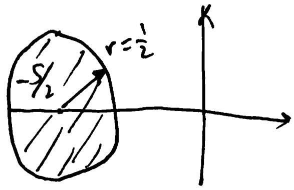
domain, open

d)



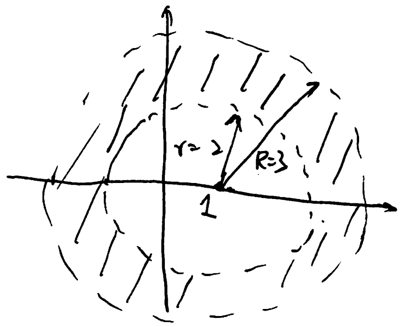
neither closed nor open

b)



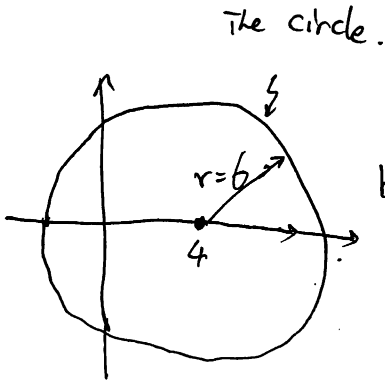
closed. bounded.

e)



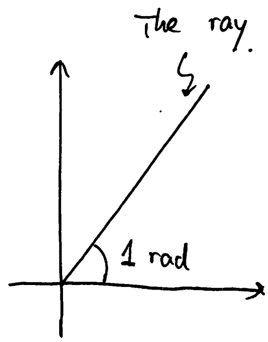
domain, open, bounded.

c).



closed bounded

f)

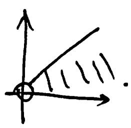


closed (If you include 0 in this set)

□

5. $W \cup \{0\}$.

Remark: W is



When showing that {accumulation points} = $W \cup \{0\}$, we need first show that for any point in $W \cup \{0\}$, ~~a sma~~ its arbitrarily small neighborhood always intersect with W with infinitely many points.

Then, we need to show that all the other points are not accumulation points, which follows from a similar argument. □.

6. Proof 1: Using geometric sequence, for $w \neq 1$, we have: ③

$$1 + \cdots + w^{n-1} = \frac{1-w^n}{1-w} = \frac{1-1}{1-w} = 0. \quad \square$$

Proof 2: Let $d = \gcd(n,k)$, where $w = e^{2\pi i k/n}$ $k \in \{1,2,\cdots,n-1\}$. | gcd = greatest common divisor.

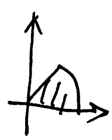
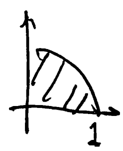
If $d=1$, then the set $\{1, w, \cdots, w^{n-1}\}$ would be exactly the same as $\{1, e^{2\pi i/n}, e^{4\pi i/n}, \cdots, e^{2(n-1)\pi i/n}\}$.

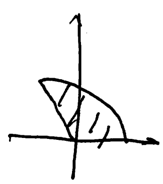
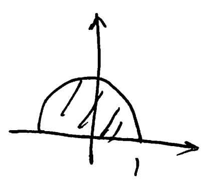
These points are equally distributed on the unit circle, and as a consequence, $1 + w + \cdots + w^{n-1} = 0$.

If $d > 1$, Then let $m = \frac{n}{d}$, then $w^m = e^{2\pi i \frac{k}{d}} = 1$.

$$\begin{aligned} 1 + \cdots + w^{n-1} = \; & 1 + w + \cdots + w^{m-1} \\ & + w^{m} + \cdots + w^{2m-1} \\ & + \cdots\cdots\cdots \\ & + w^{(d-1)m} + \cdots + w^{(d-1)m+(m-1)} \end{aligned} = d(1 + w + \cdots + w^{m-1}) = 0 \quad \square$$

from the discussion above (Noticing that w is a m-th ~~unit~~ root of unity, $\gcd(m, \frac{k}{d}) = 1$.)

7. $V =$  $s(V) =$ 

$c(V) =$  $g(V) =$ 

$\square$

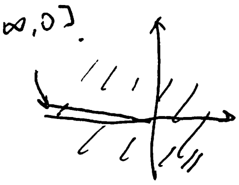
8. <u>Proof</u>: A basic observation is that the modulus function

$$r = r(z) = |z|, \quad \text{is continuous for all } z \in \mathbb{C}.$$

and the argument function

$$\theta = \mathrm{Arg}(z) \quad \text{is continuous on} \quad \mathbb{C} \setminus (-\infty, 0]$$

cont. outside $(-\infty, 0]$.



a). So $S(z)$ is continuous near i, ~~so~~ then

$$\lim_{z \to i} S(z) = S(i) = \sqrt{r(i)}\, e^{i\theta(i)/2} = \sqrt{1}\, e^{i\frac{\pi/2}{2}} = e^{i\pi/4} = \frac{1+i}{\sqrt{2}}.$$

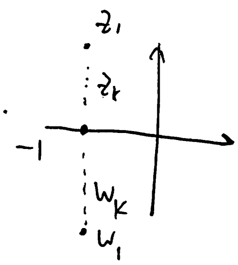
b). Using the ε-δ language to prove this statement.

For any $\varepsilon > 0$, choose $\delta = \varepsilon^2$, then for all $|z| < \delta$, we have

$$|S(z) - 0| = |S(z)| = \left|\sqrt{r}\, e^{i\theta/2}\right| = \sqrt{r} < \sqrt{\delta} = \varepsilon,$$

This shows that $\lim_{z \to 0} S(z) = 0$.

c).



$\lim_{z \to -1} r(z) = 1$ is well-defined.

If we choose a sequence $\{z_k\}$ converging to -1 ~~for~~ from the second quadrant, we have

$$\lim_{k\to\infty} \arg(z_k) = \pi, \quad \text{and then} \quad \lim_{k\to\infty} S(z_k) = e^{\pi i/2} = i.$$

On the other hand, if we choose $\{w_k\}$ converging to -1 from the third quadrant, we have $\lim_{k\to\infty} \arg(w_k) = -\pi$ and then $\lim_{k\to\infty} S(w_k) = e^{-\pi i/2} = -i$.

If $\lim_{z\to -1} S(z)$ exists, then these two limit should coincide, which leads to a contradiction.

$\square$

Remark to problem 8:

Here are some common mistakes that I found when I graded the homework.

1. By simplifying $S(z) = \sqrt{r}\, e^{i\theta/2} = \sqrt{z}$.

The problem is, in general, $\sqrt{z}$ is NOT a WELL-DEFINED function for COMPLEX numbers. There would be no ambiguity if we write $\sqrt{a}$ for POSITIVE REAL numbers $a \geq 0$. However, we DO need to define the argument function $\arg(z)$ first in order to define "$\sqrt{z}$".

Otherwise, strange thing may happen:

$$1 = \sqrt{1} = \sqrt{e^{2\pi i}} = e^{\frac{2\pi i}{2}} = e^{\pi i} = -1 \quad !!!$$

2. Some students write $\lim_{z\to 0} S(z) = S(0) = 0$ (*)

or $\lim_{z\to 0} S(z) = \lim_{z\to 0} \sqrt{z}$. (**)

The problem of (*) is that $S(0)$ is NOT DEFINED in the problem. And if you want to define $S(0)$ to be 0, and using the continuity argument, this is invalid because this question itself is asking for a proof for the continuity at 0.

The problem of (**) is ~~that~~ the same as 1.