MAT342 Homework 1

Due Wednesday, February 6

1. Simplify each of the following to the form $x+iy$ with $x,y \in \mathbb{R}$.

(a) $\dfrac{1+2i}{3-4i}+\dfrac{1}{5i}$

(b) $(1+i)^4$

(c) $\dfrac{\overline{1+i}}{|1+i|}$

2. Show that for any complex number z, $\sqrt{2}|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z|$.

3. Verify by direct calculation that $z = 2+i$ is a root of the polynomial $f(z) = z^3 - 5z^2 + 9z - 5$; that is, confirm that $f(2+i) = 0$.

4. Use Cardano's method to find a solution to the equation

$$z^3 + 3z^2 - 3z - 11 = 0.$$

(a) First, rewrite the equation in the form[†]

$$x^3 = px + q$$

by making the substitution $z = x - 1$ so that the resulting cubic has its inflection point at $x = 0$ instead of $z = 1$.

(b) Next, set $x = u + v$ so that the left-hand side can be written in the form

$$(u+v)^3 = u^3 + 3uv(u+v) + v^3 = (3uv)x + (u^3 + v^3).$$

By equating coefficients of x, observe that the two sides will be equal when

$$3uv = p \quad \text{and} \quad u^3 + v^3 = q.$$

Solving this for v gives $u^3 + (p/3u)^3 = q$, which can be rewritten as a quadratic equation in u^3 by clearing the denominator. Use the quadratic formula (or just factor it) to solve the resulting equation for u^3 (you will, of course, get two solutions).

(c) By symmetry, we obtain the same values for v^3, so one of the solutions in the previous part is u^3, the other is v^3. Thus, a solution to the equation in part (a) will be $x = u + v$, that is, the sum of the cube roots of the two solutions in part (b).

(d) Finally, the solution to the original equation can be found as $z = x - 1$.

(You should get $z = 2^{2/3} + 2^{1/3} - 1$ as your answer.)

[†] In class, I wrote this as $x^3 = 3px + 2q$ to simplify the resulting formula.