MAT342 Homework 1

Solutions

1. Simplify each of the following to the form $x+iy$ with $x,y \in \mathbb{R}$.

(a) $\dfrac{1+2i}{3-4i}+\dfrac{1}{5i} \quad = \dfrac{(1+2i)(3+4i)}{9+16}-\dfrac{i}{5}=\dfrac{-1+2i}{5}-\dfrac{i}{5}=\dfrac{-1+i}{5}.$

(b) $(1+i)^4 \quad = \left(\sqrt{2}e^{i\pi/4}\right)^4 = 4e^{i\pi} = -4.$

(c) $\dfrac{\overline{1+i}}{|1+i|} \quad = \dfrac{1-i}{\sqrt{2}}.$

2. Show that for any complex number z, $\sqrt{2}|z| \geq |\mathrm{Re}\, z| + |\mathrm{Im}\, z|$.

For any $\theta \in \mathbb{R}$, it is easy to see that

$$1 \leq |\cos\theta| + |\sin\theta| \leq \sqrt{2}$$

using basic calculus. Multiply each term of the above inequality by $|z|$ where $\arg z = \theta$, and simplify using $|z|\cos\theta = \mathrm{Re}\, z$ and $|z|\sin\theta = \mathrm{Im}\, z$ to obtain

$$|z| \leq |\mathrm{Re}\, z| + |\mathrm{Im}\, z| \leq \sqrt{2}|z|.$$

3. Verify by direct calculation that $z=2+i$ is a root of the polynomial $f(z)=z^3-5z^2+9z-5$; that is, confirm that $f(2+i)=0$.

Observe that $(2+i)^2 = 3+4i$ and $(2+i)^3 = 2+11i$. Then we have

$$\begin{aligned} f(2+i) &= (2+11i)-5(3+4i)+9(2+i)-5 \\ &= (2-15+18-5)+(11-20+9)i = 0+0i = 0. \end{aligned}$$

4. Use Cardano's method to find a solution to the equation

$$z^3+3z^2-3z-11=0.$$

(a) First, rewrite the equation in the form†

$$x^3 = px+q$$

by making the substitution $z=x-1$ so that the resulting cubic has its inflection point at $x=0$ instead of $z=1$.

Letting $z=x-1$, we obtain

$$\begin{aligned} z^3+3z^2-3z-11 &= (x^3-3x^2+3x-1)+3(x^2-2x+1)-3(x-1)-11 \\ &= x^3+(-3+3)x^2+(3-6-3)x+(-1+3+3-11) \\ &= x^3-6x-6. \end{aligned}$$

Since the second derivative of the above cubic in x is $6x$, the inflection point is at 0, so it is "centered". We can rewrite the equation for the roots as

$$x^3 = 6x+6.$$

†In class, I wrote this as $x^3 = 3px+2q$ to simplify the resulting formula.

(b) Next, set $x = u + v$ so that the left-hand side can be written in the form

$$(u+v)^3 = u^3 + 3uv(u+v) + v^3 = (3uv)x + (u^3 + v^3).$$

By equating coefficients of x, observe that the two sides will be equal when

$$3uv = p \quad \text{and} \quad u^3 + v^3 = q.$$

Solving this for v gives $u^3 + (p/3u)^3 = q$, which can be rewritten as a quadratic equation in u^3 by clearing the denominator. Use the quadratic formula (or just factor it) to solve the resulting equation for u^3 (you will, of course, get two solutions).

As suggested, writing $x = u + v$ gives us

$$x^3 = (u+v)^3 = (3uv)x + (u^3 + v^3) = 6x + 6,$$

so we want to find u and v for which

$$3uv = 6 \qquad \text{and} \qquad u^3 + v^3 = 6.$$

The right-hand equation gives $v = \frac{6}{3u} = 2/u$, and substituting it on the left yields

$$u^3 + \frac{8}{u^3} = 6 \qquad \text{or, equivalently,} \qquad (u^3)^2 - 6(u^3) + 8 = 0.$$

This can be factored as $(u^3 - 4)(u^3 - 2) = 0$, and hence u must satisfy either $u^3 = 2$ or $u^3 = 4$, that is, $u = 2^{1/3}$ or $u = 4^{1/3} = 2^{2/3}$.

(c) By symmetry, we obtain the same values for v^3, so one of the solutions in the previous part is u^3, the other is v^3. Thus, a solution to the equation in part **??** will be $x = u + v$, that is, the sum of the cube roots of the two solutions in part **??**.

As it says above, if $u = 2^{1/3}$, we must have $v = 2^{2/3}$. Since $x = u + v$ is a solution to the intermediate equation $x^3 = 6x + 6$, it must be that

$$\left(2^{1/3} + 2^{2/3}\right)^3 = 6(2^{1/3} + 2^{2/3}) + 6,$$

which you can check if you like.

(d) Finally, the solution to the original equation can be found as $z = x - 1$.
(You should get $z = 2^{2/3} + 2^{1/3} - 1$ as your answer.)

Right, like that. $z = x - 1 = 2^{2/3} + 2^{1/3} - 1$ solves the original equation.

Note that once you have that, you can find the other two roots of the cubic (which are complex for this particular equation) by dividing out by $z - 2^{2/3} - 2^{1/3} + 1$ to obtain the quadratic equation

$$z^2 + (2 + 2^{2/3} + 2^{1/3})z + (2 \cdot 2^{2/3} + 3 \cdot 3^{1/3} - 1) = 0.$$

Then you can use the quadratic formula.
Of course, is just easier to ask Wolfram Alpha (or similar) to solve the equation. But Cardano had *really slow* internet, so he couldn't do that.