

25. (expires 4/12) The (unforced) **Van der Pol equation** can be written as  $x'' = \mu(1 - x^2)x' - x$ . This system was originally devised in the 1920s to describe behavior in electrical circuits that use vacuum tubes, but has many other applications including **modeling the action potential of neurons**. Letting  $x' = y$  yields the system

$$\left\{ \frac{dx}{dt} = y, \quad \frac{dy}{dt} = \mu(1 - x^2)y - x \right\}.$$

For all values of  $\mu$ , the origin is a fixed point.

Show (by examining the linearization) that for  $\mu = 0$  the origin is a center, but for  $\mu > 0$  the origin is a source (a spiral source for  $0 < \mu < 2$ , although solutions near  $(0, 0)$  rotate clockwise for all  $\mu > 0$ ).

However, for  $\mu > 0$  solutions starting far away from the origin tend towards it, resulting in a **limit cycle** (this is a periodic solution which is not a fixed point, but corresponds to an oscillatory solution; nearby solutions tend towards it).

Use **DEplot** to illustrate the behavior of solutions for  $\mu = 0$ ,  $\mu = 0.5$ , and  $\mu = 2$ , making plots in both the  $(x, y)$ -plane and for  $x$  as a function of  $t$ , choosing several initial conditions and coloring the corresponding solutions differently so they can be distinguished.

I suggest defining the system as a function of  $\mu$  to minimize typing and typos. Looking at  $-5 < x < 5$ ,  $-5 < y < 5$  is a good range (when plotting  $x$  vs  $y$ , use the **scaling=constrained** option). A good selection of initial conditions has  $x(0)$  ranging from  $-5.1$  to  $4.9$  stepping by  $1$  and  $y(0) = 0$ , but let  $-4 < t < 20$ . Also using **obsrange=false** is a good idea here, and you should decrease the stepsize for more accurate solution curves. Using something like `display( DEplot(...) | DEplot(..., scene=[t,x]) )` is a good way to show the two plots side by side.

Maple will likely complain about encountering some singularities. If these messages annoy you, you can shut them up with `interface(warnlevel=0)`. Be aware this suppresses all warning messages until you reset it (the default is `warnlevel=3`).

26. (expires 4/12) The **Lorenz system** is a system of three differential equations devised in the early 1960s as a model of convection in the atmosphere, and are written as

$$\left\{ \frac{dx}{dt} = \sigma(y - x), \quad \frac{dy}{dt} = x(\rho - z) - y, \quad \frac{dz}{dt} = xy - \beta z \right\}.$$

With the “classic” values  $\sigma = 10$ ,  $\beta = 8/3$ , and  $\rho = 28$ , this system has three fixed points.

Find the three fixed points, and then compute the eigenvalues of the linearization of the system at each of them. What does this tell you about behavior of solutions near each of the fixed solutions?

27. (expires 4/12) The Lorenz equations given in **Exercise 26** above give rise to a well-known chaotic attractor: almost all initial conditions result in solutions which are quickly attracted to a butterfly-like set, but these solutions are chaotic in the sense that they are aperiodic and have sensitive dependence to initial conditions. That is, arbitrarily small changes in initial conditions will cause the solutions to have drastically different futures.



The Maple command **DEplot3d** is similar to **DEplot**, but enables production of a 3-dimensional plot of solutions to a system of ODEs. Use **DEplot3d** to produce a three-dimensional plot of the Lorenz Attractor, using  $\sigma = 10$ ,  $\beta = 8/3$ , and  $\rho = 28$ .

I suggest using  $-30 < x < 30$ ,  $-30 < y < 30$ , and  $0 < z < 52$  with **scaling=constrained**. I also used  $0 < t < 75$  with **stepsize=0.01**, **method=dverk78**, **linecolor=t**, **thickness=1** to produce the image above, which is part of a single solution curve for the system with the color indicating the time. Your picture will differ a bit, depending on what initial condition you start with, but should retain the “butterfly” shape.