22. (*expires 3/29*) In this problem will study the Lotke-Volterra predator-prey equations. In a very simple ecosystem there are two populations, whose numbers at a time t (with t in, say, years) are given by $f(t)$ (foxes) and $r(t)$ (rabbits). The evolution of these quantities obeys the system

$$\begin{cases} \dot{f}(t) &=& G_f\, f(t) + E\, f(t)\, r(t), \\ \dot{r}(t) &=& G_r\, r(t) - E\, f(t)\, r(t); \end{cases}$$

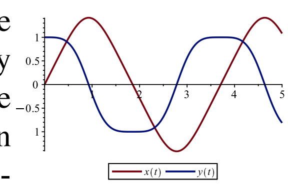
where G_f and G_r are the growth rates for the foxes and the rabbits, respectively, in the absence of each other. E is the probability of a fatal encounter between a fox and a rabbit (normalized per number of foxes and rabbits).

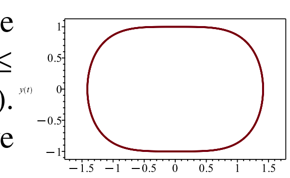
First, write some words to explain why these equations make sense. Then, fix $G_f = 0.4$, $G_r = 2.4$ (it's notorius that rabbits have the tendency to reproduce quickly) and $E = 0.01$. For a few initial conditions of your choice, plot the trajectories in the (f, r)-plane (say, with $0 \le f \le 1000$ and $0 \le r \le 1000$). For the same initial conditions, plot the actual solutions too (i.e, $f(t)$ against t, and $r(t)$ against t). Write some comments interpreting how the behavior of the solutions relates to what happens to the two species. (Here, to plot $f(t)$ against t, you can use the `scene` argument to `DEplot`, or you can use `dsolve` and maybe `plots[odeplot]`.)

Finally, repeat the same procedure with $G_f = -1.1$. Things change substantially. As above, explain how the solution behavior relates to the populations of foxes and rabbits. What does having $G_f = -1.1$ mean in the context of rabbit and fox populations?

23. (*expires 3/29*) In class on March 7, we discussed various numerical methods for solving differential equations: For a given stepsize h, a numerical method takes a (possibly approximate) solution at time t and samples the derivatives at some number of values between t and $t + h$ to get an approximate solution at time $t+h$. If the error in the approximation is ϵ, the **order** of the method is k if $\epsilon = O(h^k)$, that is, ϵ shrinks faster than ch^k some constant c as $h \to 0$.

The system $\{x' = 2y, y' = -x^3\}$ can be solved exactly using elliptic integrals. The initial condition $\{x(0) = 0, y(0) = 1\}$ gives a solution of period approximately $T_p \approx 3.708149354602744$. Using the `numeric` option to `dsolve`, demonstrate that Euler's method (`method=classical[foreuler]`) is of order 1, the Heun formula (`method=classical[heunform]`) is of order 2, and 4th-order Runke-Kutta (`method=classical[rk4]`) is of order 4 by comparing the approximate solution $x(T_p)$ to the exact one ($x(T_p) = 0$) for various values of h with $0.001 \le h \le 0.2$ for each of these methods. Use at least 10 values of h (more is better). A graph of error vs h would be nice, but not required if you explain well. Leave out values of h yielding an error greater than 0.1 – this is huge in this context.





24. (*expires 3/29*) Consider the vector field $\mathbf{F}(x, y) = \left\langle -x(x^4 + y^4) - y\,,\, x - y(x^4 + y^4) \right\rangle$.
Use Maple to draw the either the direction field or the vector field, together with some well-chosen solution curves. (I would use `DEplot` here, but you can use a combination of `fieldplot`, `dsolve` (with the `numeric` option), `plots[odeplot]`, and `plots[display]` if you prefer.)

Then *prove* that the origin is a global attractor in the future, i.e., for every solution $\mathbf{z}(t) = \big(x(t), y(t)\big)$, we have $\lim_{t\to+\infty} \mathbf{z}(t) = \mathbf{0}$.

Note: The proof is not long, but requires a mathematical argument, not a maple calculation. The proof may depend on something you calculated in maple, but more will be needed. Polar coordinates can be your friend.