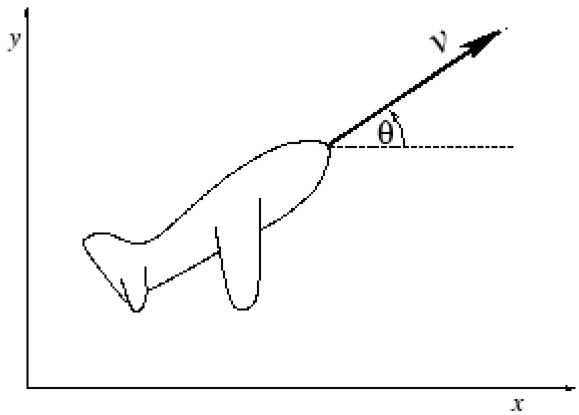
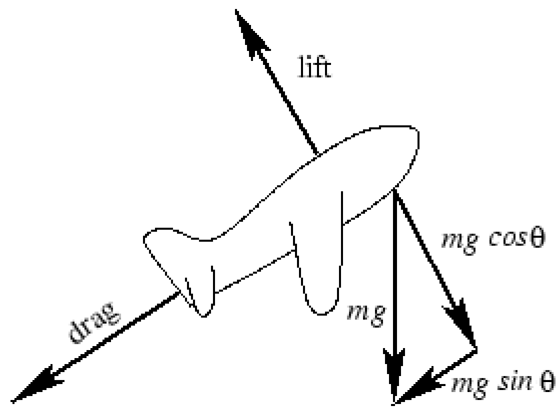
**Feb 29, 2024**

Starting to talk about a model of glider flight.

# Background about the Phugoid

The Phugoid model is a system of two nonlinear differential equations in a frame of reference relative to the plane. Let v(t) be the speed the plane is moving forward at time t, and $\theta(t)$ be the angle the nose makes with the horizontal. As is common, we will suppress the functional notation and just write v when we mean v(t), but it is important to remember that v and $\theta$ are functions of time.




If we apply Newton's second law of motion (force = mass × acceleration) and examine the major forces acting on the plane, we see easily the force acting in the forward direction of the plane is

$$\frac{m\,dv}{dt} = -mg\sin\theta - drag$$

This matches with our intuition: When $\theta$ is negative, the nose is pointing down and the plane will accelerate due to gravity. When $\theta > 0$, the plane must fight against gravity.

In the normal direction, we have centripetal force, which is often expressed as $\frac{mv^2}{r}$, where $r$ is the instantaneous radius of curvature. After noticing that that $\frac{d\theta}{dt} = \frac{v}{r}$, this can be expressed as $\frac{v\,d\theta}{dt}$, giving

$$m\,v\,\frac{d\theta}{dt} = -mg\cos\theta + lift$$

Experiments show that both drag and lift are proportional to $v^2$ and we can choose our units to absorb most of the constants. Thus, the equations simplify to the system

$$\left\{ \frac{dv}{dt} = -Rv^2 - \sin\theta,\ \frac{d\theta}{dt} = \frac{-\cos\theta + v^2}{v} \right\}$$

which is what we will use henceforth. Note that we must always have $v > 0$.

Stuff about solving differential equations now.

```
> int(x^3,x)
```

$$\frac{x^4}{4} \tag{1}$$

Can think of this as finding $f(x)$ so that $\frac{df}{dx} = x^3$

> $dsolve(f'(x) = x^3)$

$$f(x) = \frac{x^4}{4} + c_1 \tag{2}$$

```
> dsolve({diff(f(x), x) = x^3, f(1)=7})
```

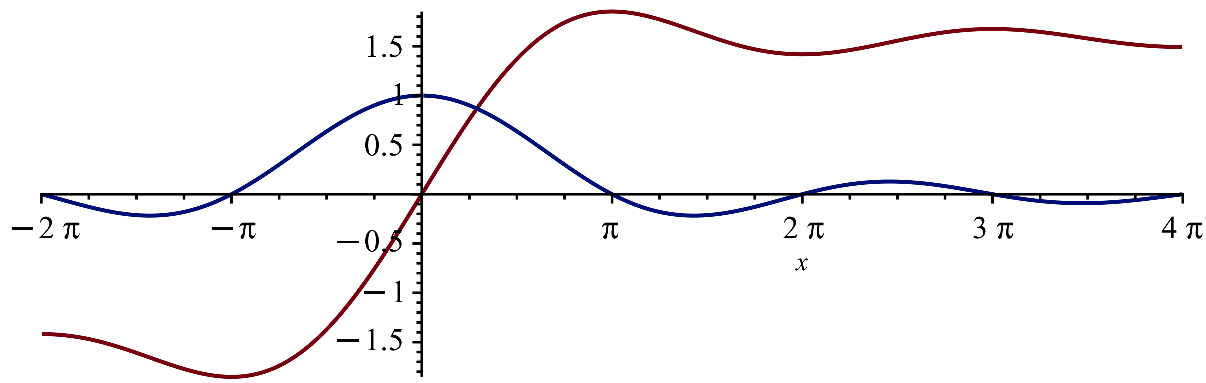
$$f(x) = \frac{x^4}{4} + \frac{27}{4} \tag{3}$$

> $int\left(\frac{\sin(x)}{x}, x\right)$

$$\mathrm{Si}(x) \tag{4}$$

Si(x) is just the integral of sin(x)/x.... duh.

```
> plot([Si(x), diff(Si(x),x)],x=-2*Pi..4*Pi);
```




> $MySol := dsolve(\{x'(t) + y(t) = t \cdot x(t), y'(t) + y(t) = t, x(0) = 0, y(0) = -2\});$

$$MySol := \left\{ x(t) = \frac{\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2} + \frac{\sqrt{2}}{2}\right)\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}+\frac{t^2}{2}}}{2} + \frac{\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}\right)\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{\frac{t^2}{2}}}{2} + 1 + \left(-\frac{\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}}{2}\right)\mathrm{e}^{\frac{1}{2}}\sqrt{\pi}}{2} - 1\right)\mathrm{e}^{\frac{t^2}{2}}, y(t) = t - 1 - \mathrm{e}^{-t} \right\} \tag{5}$$

```
> plot([MySol[1],MySol[2]], t=-2..2)
Error, (in plot) incorrect first argument [x(t) = 1/2*erf(1/2*2^(1/2)
*t+1/2*2^(1/2))*2^(1/2)*Pi^(1/2)*exp(1/2+1/2*t^2)+1/2*erf(1/2*2^(1/2)
*t)*2^(1/2)*Pi^(1/2)*exp(1/2*t^2)+1+(-1/2*2^(1/2)*erf(1/2*2^(1/2))*
exp(1/2)*Pi^(1/2)-1)*exp(1/2*t^2), y(t) = t-1-exp(-t)]
```

Why didn't this work?

```
> MySol[1]
```

$$x(t) = \frac{\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}+\frac{\sqrt{2}}{2}\right)\sqrt{2}\,\sqrt{\pi}\,e^{\frac{1}{2}+\frac{t^2}{2}}}{2} + \frac{\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}\right)\sqrt{2}\,\sqrt{\pi}\,e^{\frac{t^2}{2}}}{2} + 1 + \left(-\frac{\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}}{2}\right)e^{\frac{1}{2}}\sqrt{\pi}}{2} - 1\right)e^{\frac{t^2}{2}} \quad \textbf{(6)}$$

because MySol is a pair of equations, not functions. We can pick off the functions in either of the two following ways.

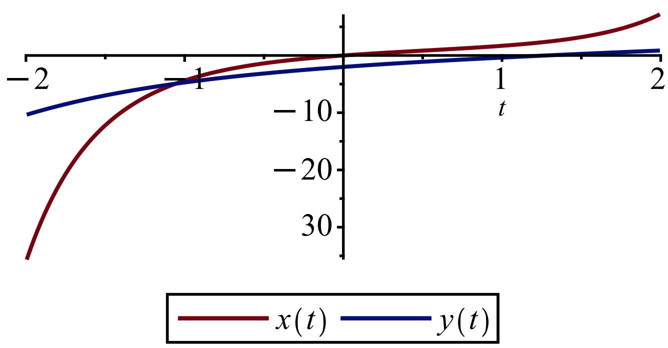
```
> rhs(MySol[1])
```

$$\frac{\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}+\frac{\sqrt{2}}{2}\right)\sqrt{2}\,\sqrt{\pi}\,e^{\frac{1}{2}+\frac{t^2}{2}}}{2} + \frac{\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}\right)\sqrt{2}\,\sqrt{\pi}\,e^{\frac{t^2}{2}}}{2} + 1 + \left(-\frac{\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}}{2}\right)e^{\frac{1}{2}}\sqrt{\pi}}{2} - 1\right)e^{\frac{t^2}{2}} \quad \textbf{(7)}$$

```
> subs(MySol,[x(t),y(t)])
```

$$\left[\frac{\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}+\frac{\sqrt{2}}{2}\right)\sqrt{2}\,\sqrt{\pi}\,e^{\frac{1}{2}+\frac{t^2}{2}}}{2} + \frac{\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}\right)\sqrt{2}\,\sqrt{\pi}\,e^{\frac{t^2}{2}}}{2} + 1 + \left(-\frac{\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}}{2}\right)e^{\frac{1}{2}}\sqrt{\pi}}{2} - 1\right)e^{\frac{t^2}{2}},\ t - 1 - e^{-t}\right] \quad \textbf{(8)}$$

```
> plot(subs(MySol,[x(t),y(t)]), t=-2..2,size=[.5,.5],legend=[x(t),y(t)]);
```




Now back to thinking about gliders.

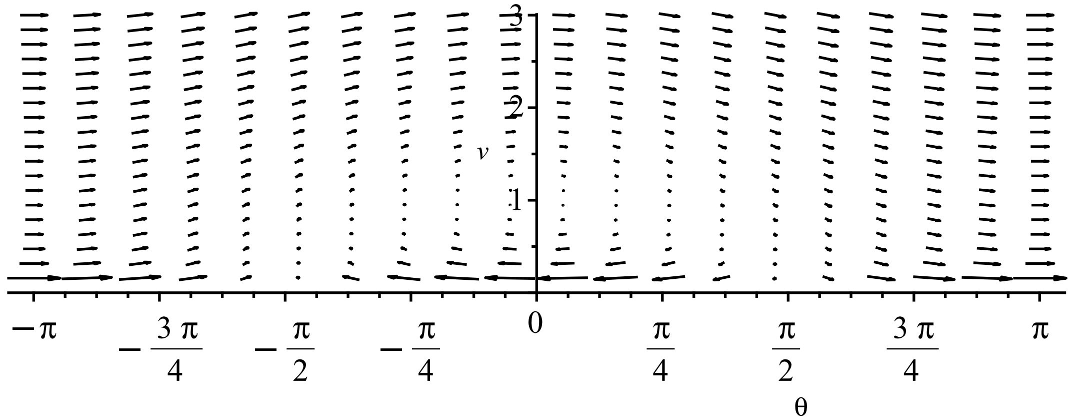
$$> \ phug := \left[\ diff(v(t), t) = -\sin(\mathrm{theta}(t)) - R\cdot v(t)^2,\ diff(\mathrm{theta}(t), t)\right.$$

$$= \frac{\left( (v(t))^2 - \cos(\text{theta}(t)) \right)}{v(t)} \Big];$$

$$phug := \left[ \frac{d}{dt} v(t) = -\sin(\theta(t)) - R\, v(t)^2, \frac{d}{dt} \theta(t) = \frac{v(t)^2 - \cos(\theta(t))}{v(t)} \right] \quad \textbf{(9)}$$

The solution, when you can find it, is horrible and uninformative. Let's make a vector field instead

```
> with(plots):
> fieldplot( [v -cos(theta)/v, -sin(theta)],
        theta=-Pi..Pi, v=0..3, arrows=slim,
        tickmarks=[piticks,default], size=[.8,.4]);
```




I'd rather have a direction field -- ie, all vectors have the same length.

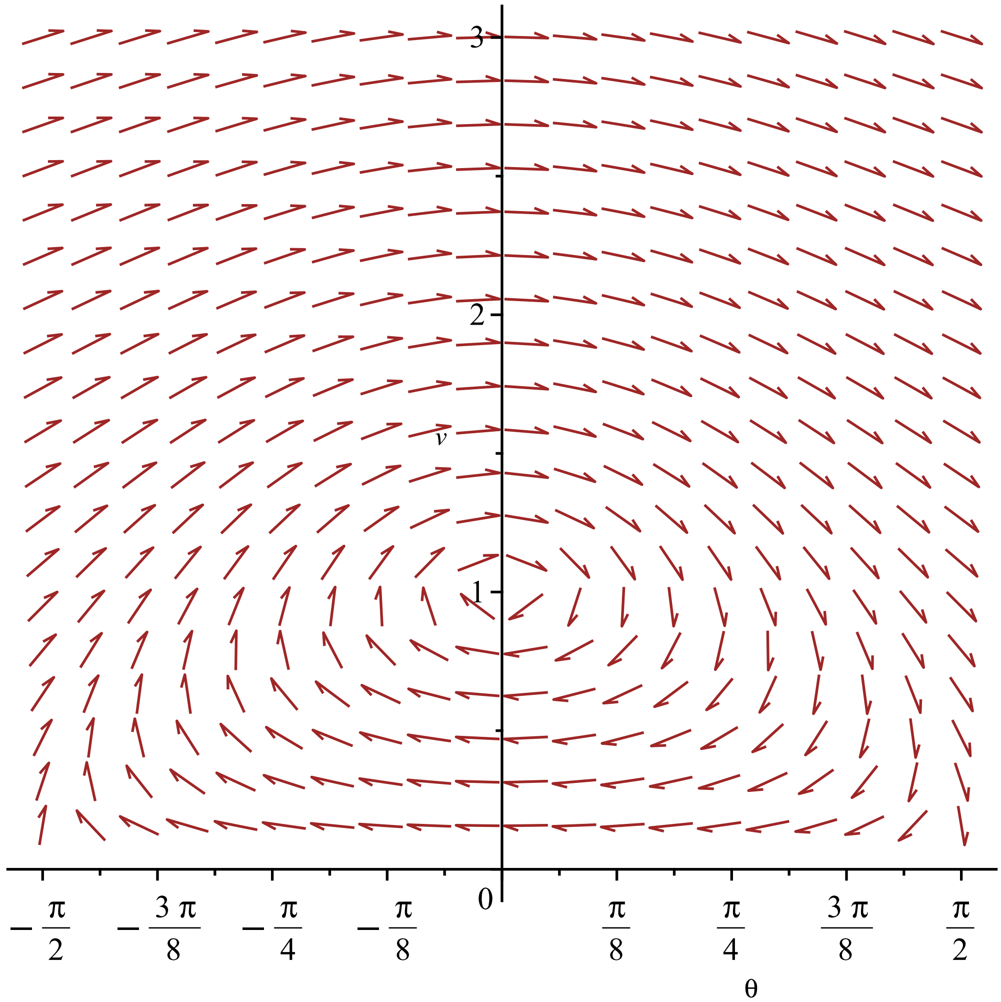
```
> with(DEtools):
> R:=0;  # let's ignore friction for now
```

$$R := 0 \quad \textbf{(10)}$$

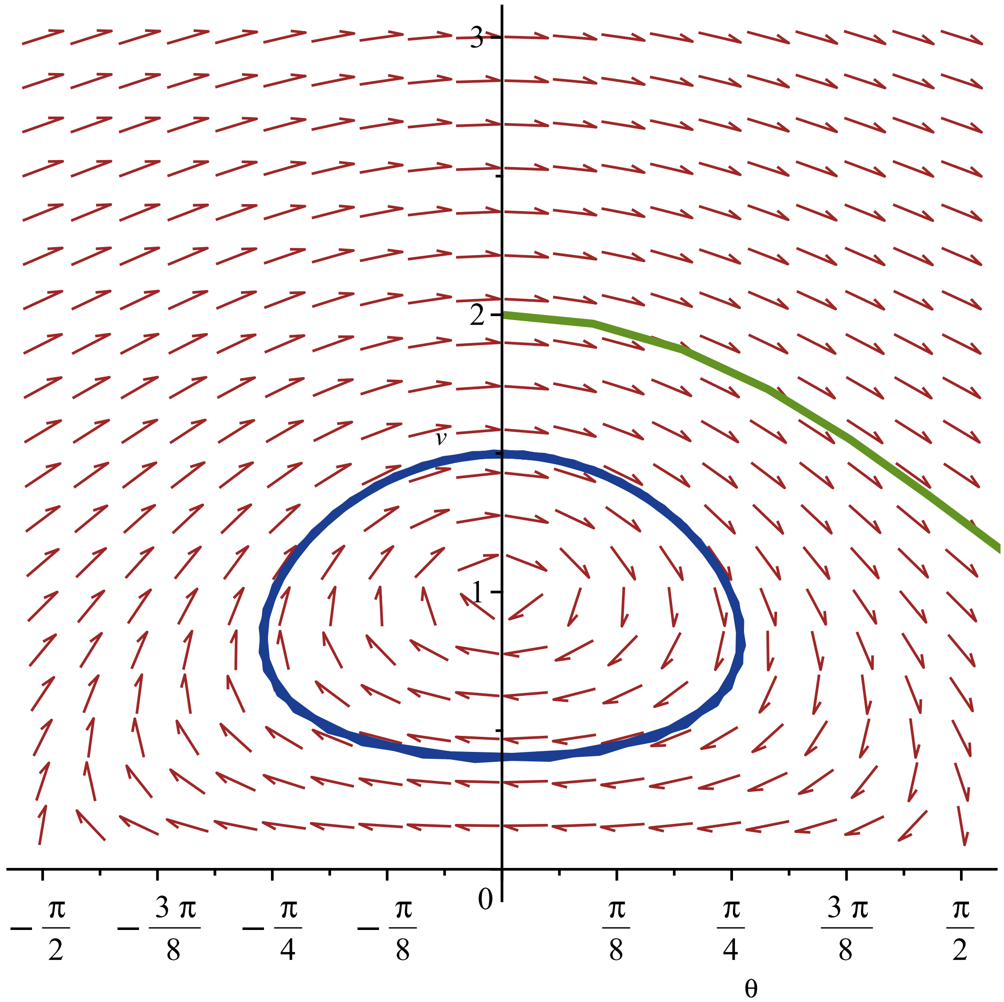
```
> phug;
```

$$\left[ \frac{d}{dt} v(t) = -\sin(\theta(t)), \frac{d}{dt} \theta(t) = \frac{v(t)^2 - \cos(\theta(t))}{v(t)} \right] \quad \textbf{(11)}$$

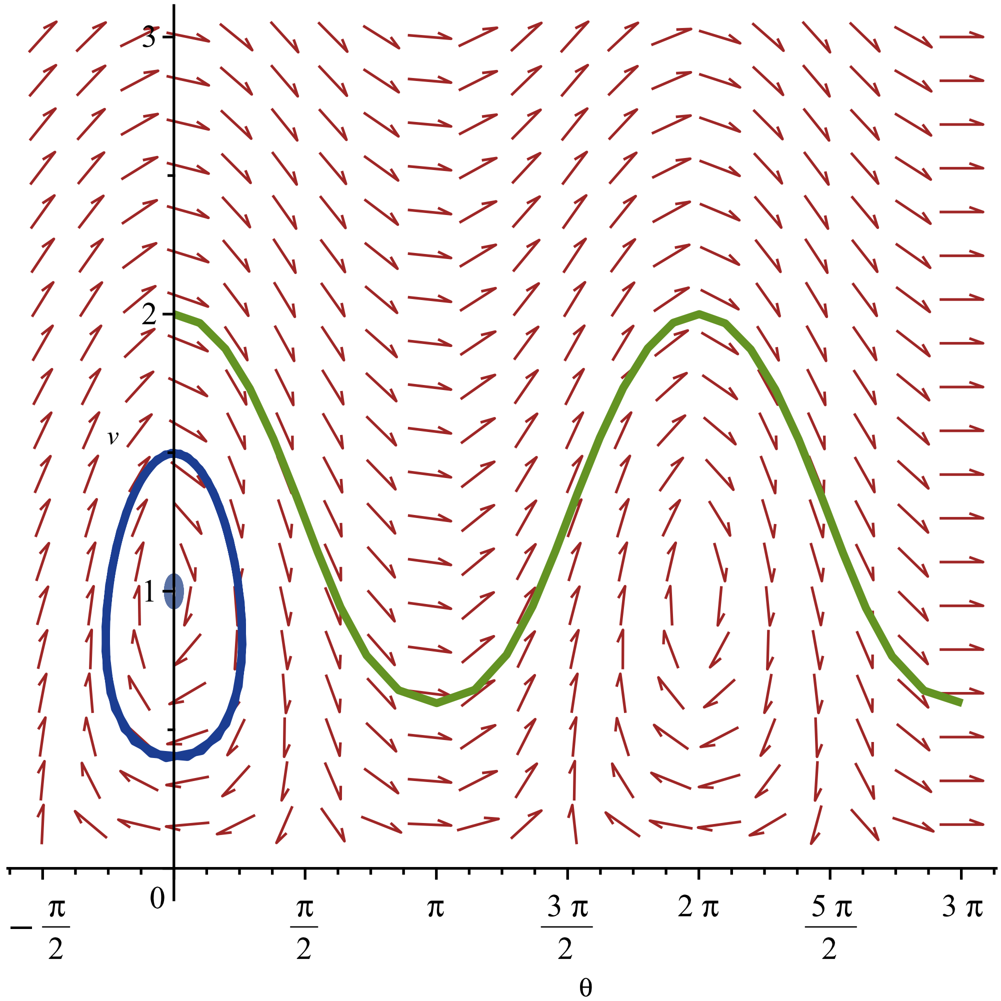
```
> DEplot(phug, [theta(t),v(t)], t=0..10, theta=-Pi/2..Pi/2,
   v=0..3, tickmarks=[piticks,default])
```



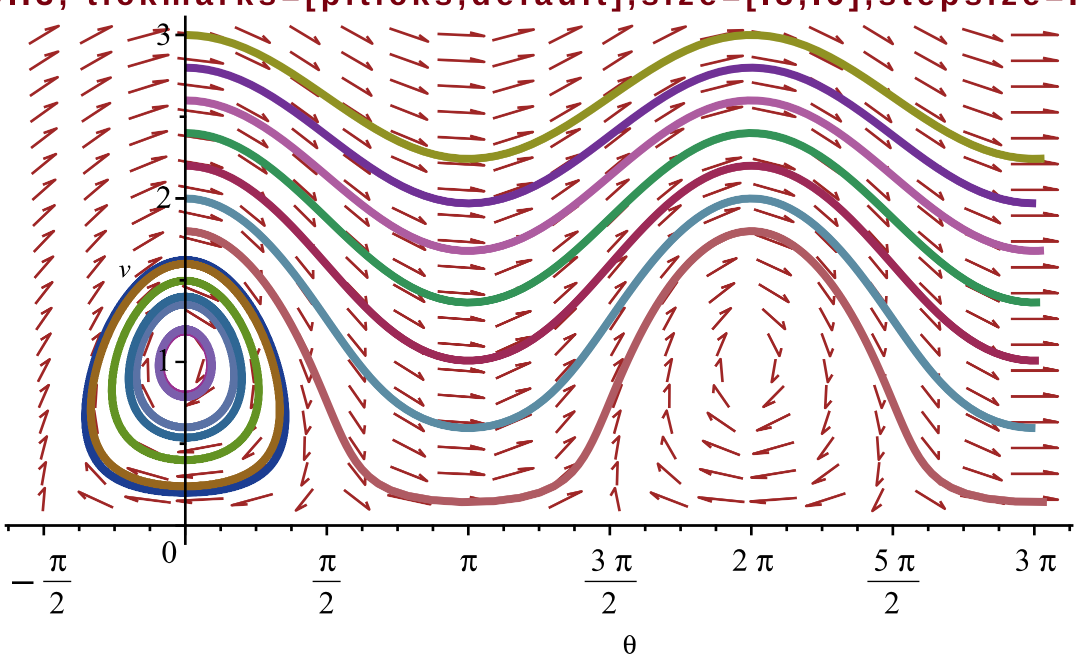

```
> DEplot(phug, [theta(t),v(t)], t=0..10,
  [[theta(0)=0, v(0)=1.5],   [theta(0)=0, v(0)=2]],
  theta=-Pi/2..Pi/2,
    v=0..3, tickmarks=[piticks,default])
```




```
> DEplot(phug, [theta(t),v(t)], t=0..10,
  [[theta(0)=0, v(0)=1.5],   [theta(0)=0, v(0)=2],
    [theta(0)=0, v(0)=1.05]],
  theta=-Pi/2..3*Pi,
    v=0..3, tickmarks=[piticks,default])
```

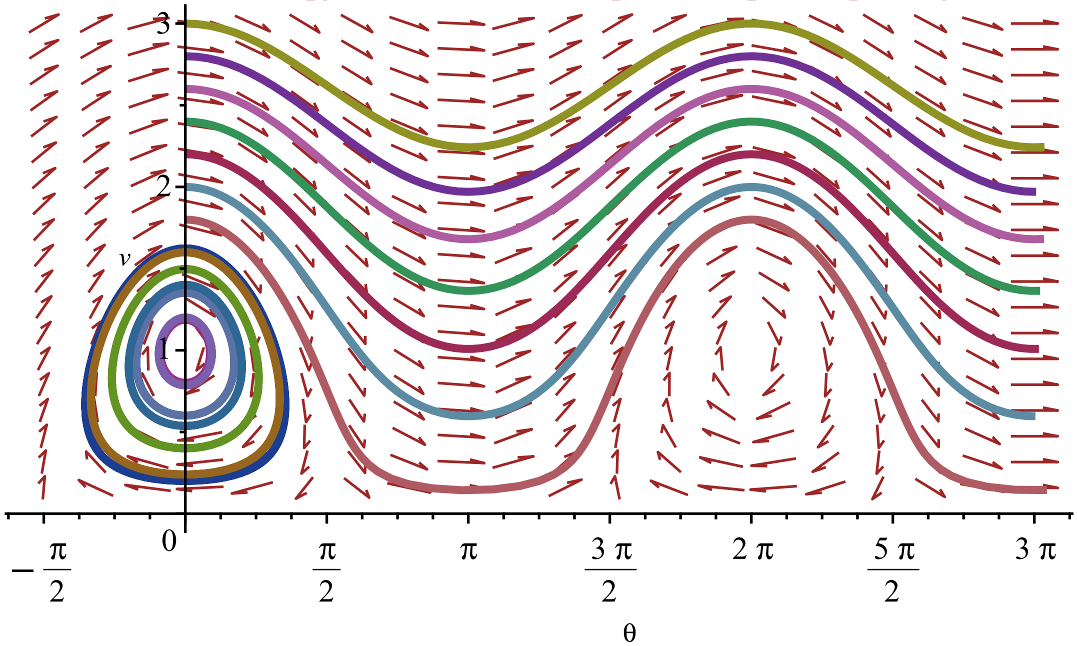



```
> DEplot(phug, [theta(t),v(t)], t=0..10,
  [seq([theta(0)=0, v(0)=.2*i],i=1..20)],
  theta=-Pi/2..3*Pi,
    v=0..3, tickmarks=[piticks,default],size=[.8,.6],stepsize=.05)
```



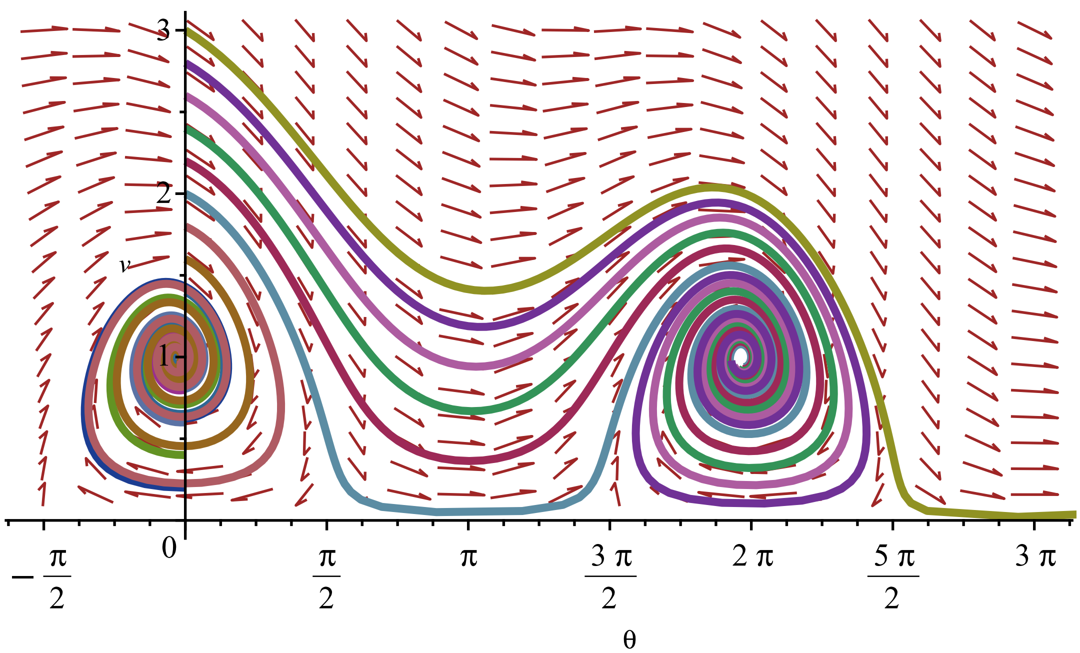

```
> DEplot(phug, [theta(t),v(t)], t=0..10,
```

```
[seq([theta(0)=0, v(0)=.2*i],i=1..20)],
theta=-Pi/2..3*Pi,
  v=0..3, tickmarks=[piticks,default],size=[.8,.6],stepsize=.05)
```




```
> R:=.1;
  DEplot(phug, [theta(t),v(t)], t=0..20,
  [seq([theta(0)=0, v(0)=.2*i],i=1..20)],
  theta=-Pi/2..3*Pi,
    v=0..3, tickmarks=[piticks,default],size=[.8,.6],stepsize=.05)
```

$$R := 0.1$$

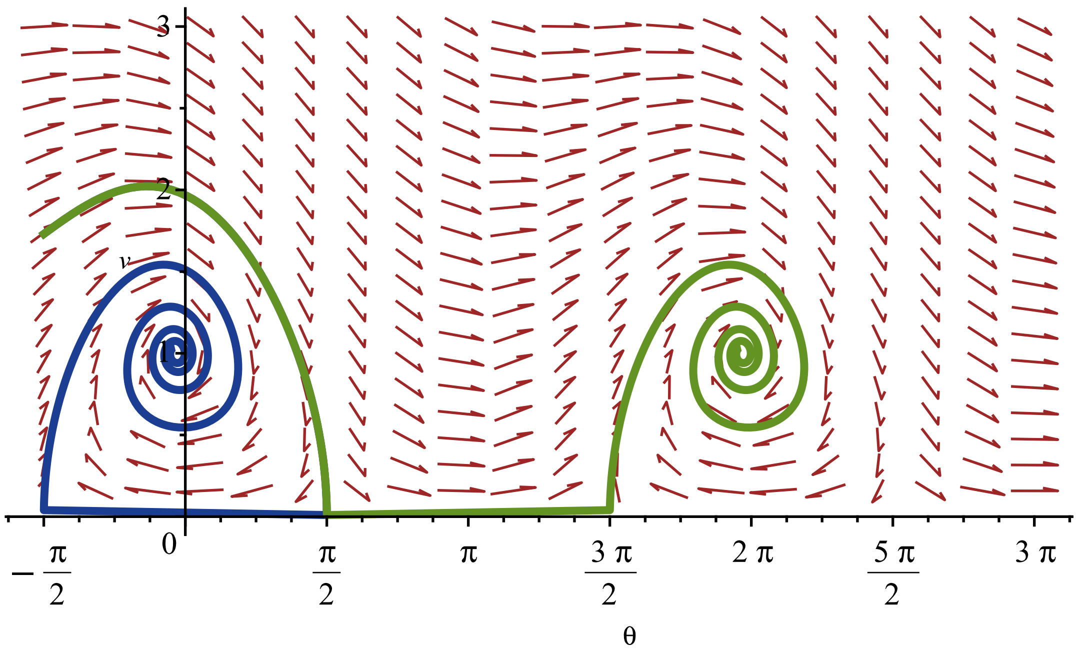



Near $\theta = \frac{\pi}{2}$, $v = 0$ , the system becomes undefined. The plane is nearly vertical and stalls. We can see this below -- a tiny change in velocity has a dramatic change in behaviour.

```
> R:=.1;  #
  DEplot(phug, [theta(t),v(t)], t=-20..20,
```

```
[[theta(0)=Pi/2-.001,v(0)=.01], [theta(0)=Pi/2+.001,v(0)=.01]],
theta=-Pi/2..3*Pi,
  v=0..3, tickmarks=[piticks,default],size=[.8,.6],stepsize=.05)
```

$R := 0.1$




```
> phug
```

$$\left[\frac{d}{dt}\, v(t) = -\sin(\theta(t)) - 0.1\, v(t)^2,\ \frac{d}{dt}\, \theta(t) = \frac{v(t)^2 - \cos(\theta(t))}{v(t)}\right] \quad (12)$$