

NOTE: Each exercise is worth 10 points and can be turned in at any time before its “expiration date”. At the end of the semester, I will expect you to have turned in at least 1/2 of the exercises assigned. If you do more, I will pick your best grades. If you do less, the missing grades will be counted as zeros. Altogether, these will count the same as one project.

Many of these problems will require you to use the help system and/or read the text to figure out what commands you need to use and how to use them.

1. (expires 2/6) If $p(x)$ is a complex polynomial with real coefficients, it is well known that it can be factored into a product of linear and quadratic terms with real coefficients, or into a product of linear terms only if the coefficients are allowed to be complex.

First, use `Maple` to write $q(z) = x^5 - 3x^4 - 3x^3 + 9x^2 - 10x + 30$ as a product of *exact* linear and quadratic factors with real coefficients. By exact, I mean you should leave any non-rational factors expressed as radicals; do not approximate terms like $\sqrt{3}$ as 1.73205, etc.

Then write $q(x)$ as a product of only linear factors (which will involve complex numbers).

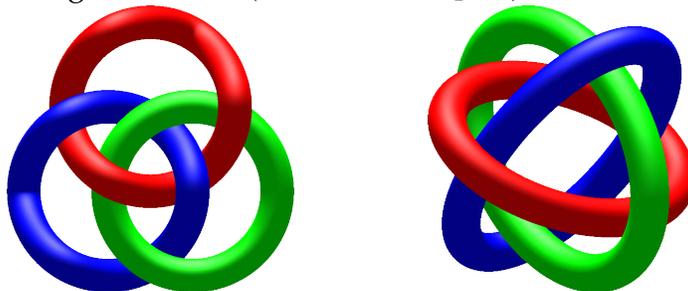
Finally, do the same for product $p(x) = x^5 - 2x^4 - 10x^3 + 20x^2 - 16x + 32$.

Hint: Note that this question asks for four different answers, two for each polynomial. While the maple command `factor` is relevant, it will need a little assistance to be able to answer all four parts. Alternatively, there are other ways to do this.

2. (expires 2/6) Draw a graph showing both $\cos(x)$ and its fifth Taylor polynomial (that is, $1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4$) for x between -4 and 4 . What degree of Taylor polynomial seems to be needed to get good agreement in this range? *Hint:* use a variation of the command `convert(taylor(cos(x), x, 5), polynomial)` to make this work. Think of a suitable way to demonstrate that the approximation you have taken is “good” – what is a good definition of “good” here? Be sure to **explain** your choice of the meaning of “good”.
3. (expires 2/13) The Borromean rings are a set of three mutually interlocked rings, arranged so that if one ring is removed, the other two become unlinked.

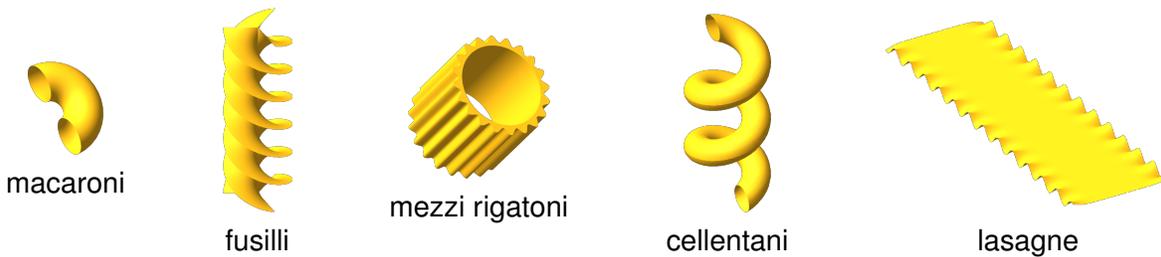
Use `Maple` to make an image of the Borromean rings in \mathbb{R}^3 . *Hint:* I suggest using `tubeplot` from the `plots` package. The real challenge of this problem is figuring out how to parameterize the three curves.

Note that the Borromean rings can not be made from flat, round rings. Either some of the rings have to be bent a little to weave through each other, or flat rings which are elliptical can be used. See the figures below (made with `Maple`).



4. (expires 2/13) Plot the function $f(x) = 2 \sin x - x^3 - 1/5$, for $x \in [-4, 4]$. Find all the zeros of the function with an accuracy of 20 decimal digits. *Hint:* See `Digits`, `fsolve`.

- (expires 2/20) Consider the planar curve γ defined by $x^2y^3 + y^2 + y - 2e^x = 0$. Using `Maple`, find the slope of the tangent line to the curve at $(0, 1)$. Then plot the curve and the tangent line on the same graph.
Hint: you might want to use `implicitplot` from the library `plots`. You might find `implicitdiff` helpful, too.
- (expires 2/20) Define a `Maple` function g that, given a positive integer k yields the sum of the first k primes. What is k such that $g(k) \leq 100,000$ but $g(k+1) > 100,000$? You might find `sum` and `ithprime` helpful.
- (expires 2/20) Use the Maclaurin series for $\arctan x$ (that is, the Taylor series about $x = 0$) evaluated at $x = 1/\sqrt{3}$ to compute the value of π to 30 places. How many terms are needed to compute the value to 50 places?
- (expires 2/20) Use `Maple` to make pictures of the following pasta.



Here are some relevant equations, in no particular order.

$$z = \sin(2y) \left(1 - e^{-(x/6)^8}\right) \quad -6 \leq x \leq 6, \quad -20 \leq y \leq 20$$

$$\tau = 1 \quad 0 \leq \phi \leq \pi, \quad -\pi \leq \sigma \leq \pi \quad (\text{toroidal coordinates})$$

$$x = \left(1 + \frac{\cos(s)}{2}\right) \cos(t) \quad y = \left(1 + \frac{\cos(s)}{2}\right) \sin(t) \quad z = 0.4t + \frac{\sin(s)}{2} \quad \begin{array}{l} 0 \leq s \leq 2\pi \\ \frac{\pi}{2} \leq t \leq \frac{11\pi}{2} \end{array}$$

$$\left\{ \begin{array}{l} x = r \sin(t) \quad y = r \cos(t) \quad z = t/2 \\ x = r \sin\left(t + \frac{2\pi}{3}\right) \quad y = r \cos\left(t + \frac{2\pi}{3}\right) \quad z = t/2 \\ x = r \sin\left(t - \frac{2\pi}{3}\right) \quad y = r \cos\left(t - \frac{2\pi}{3}\right) \quad z = t/2 \end{array} \right. \quad \begin{array}{l} 0 \leq r \leq 1 \\ 0 \leq t \leq 4\pi \end{array}$$

$$6 \leq r \leq 7 + \sin(20\theta)/2, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq 14 \quad (\text{cylindrical coordinates})$$

To help you get started, the `Maple` worksheet called [pasta.mw](#) draws Mezzi Rigatoni. For full credit, your pasta should look like pasta, with appropriate coloring, viewpoint, smoothness, and lighting. Sauce is optional.

9. (expires 2/27) Fit the set of points

$$(1.02, -4.30), (1.00, -2.12), (0.99, 0.52), (1.03, 2.51), (1.00, 3.34), (1.02, 5.30)$$

with a line, using the least square method we used in class. You will see that this is not a good fit. Think of a better way to find a line which is a good fit and use Maple to do it. Explain in your solution why you think your better way is indeed better.

You can load these data points from the web at <http://www.math.sunysb.edu/~scott/mat331.spr13/problems/prob9.txt>, which defines a list `prob9` containing these points.

See the page on [loading a file from the web](#) from the class web page for more details.

10. (expires 2/27) The file <http://www.math.sunysb.edu/~scott/mat331.spr13/problems/prob10.txt> defines 21 data points that approximates an exponential curve of the form $y = ae^{bx}$. This data is called `prob10`.

Find a and b by taking the appropriate logarithm, then use `LeastSquares` to find the resulting “best” line. Transform this line appropriately to get an exponential curve.

Plot the exponential and the `prob10` data on the same axes.

11. (expires 2/27) Fit the points $(-1.9, -4.7)$, $(-0.8, 1.2)$, $(0.1, 2.8)$, $(1.4, -1.2)$, $(1.8, -3.5)$ by means of a quadratic function $f(x) = ax^2 + bx + c$, using the least square method.

You can load these data points from the web at <http://www.math.sunysb.edu/~scott/mat331.spr13/problems/prob11.txt>, which defines a list `prob11` containing these points.

See the page on [loading a file from the web](#) from the class web page for more details.

12. (expires 2/27) In this problem we will estimate the charge of the electron.

If an electron of energy E is thrown into a magnetic field B which is perpendicular to its velocity, the electron will be deflected into a circular trajectory of radius r . The relation between these three quantities is:

$$B r e = \frac{E^2}{m^2 c^4} \sqrt{E^2 - m^2 c^4}, \quad (1)$$

where e and m are, respectively, the charge and the mass of the electron, and c is the speed of light. The rest mass of the electron is defined as $E_0 = mc^2$, and is about equal to $8.817 \cdot 10^{-14}$ Joules. In our experimental set-up the energy of the emitted electrons is set to be $E = 2.511E_0$.

The file `electron.txt` defines a list called `electron`. Each element of the list is a pair of the form $[B_i, r_i]$, and these quantities are expressed in Teslas and meters. Use least square fitting to determine the best value for e .

[Hint: Notice that the right hand side of eqn (1) is just a constant—calculate it once and for all and give it a name. Then eqn (1) becomes an equation which is linear in the unknown e . To verify your solution: $e \approx 1.602 \cdot 10^{-19}$ Coulomb]. Physical constants courtesy of N.I.S.T.

13. (*expires 3/6*) The text below was encrypted with a substitution cipher. Only the letters (both upper-case and lower-case) were substituted, leaving punctuation and spaces alone. Figure out what the original message was.

"wA'r aBeD WUeK AP XNaB NM U rALKNP USUeAJBMA NM zUM nPrB ZNAW
U JUM ZWP'r XBUEMNMO AP SXUD AWB aNPXNM." yWUA'r ZWUA rWB APXK
AWB SPXNCB ZWBM rWB WUMKBK AWBJ AWB BJSAD eBaPXaBe.
lNCWUeK heULANOUM, "yWB zCUeXUAAN yNXA"

If you wish, you can find the encrypted text on the class web page at <http://www.math.sunysb.edu/~scott/mat331.spr13/problems/subscript.txt>.

Hint: you might find `CountCharacterOccurrences` from the library `StringTools` helpful.

14. (*expires 3/6*) The cryptography chapter in the notes is called "fsqFsHn sGGousG", which is actually the result of applying a Caesar cipher to its original title. A 53-character alphabet consisting of all the upper-case letters, a space, and all the lower-case letters was used; consequently the space in the middle might or might not correspond to a space in the title. Determine what the original title was.

15. (expires 3/15) The string below was encrypted using an affine cipher on the 27 letter alphabet " abcdefghijklmnopqrstuvwxyz" (there is a space in the 0th position.) Decrypt it.

fmw segjaweoouanerj a ceyqrype aswaheoaqbrqabeafrua eeaojerf afmjeayperjpu

Hint: this phrase follows the the typical pattern in English where there are (almost) as many spaces as words (and so spaces are very common), and the letter "e" is also very common. You can use the technique described in chapter 4 of the notes, section 7.3.

If you wish, the encrypted text can be loaded from the file `afftext.txt` in the [problems section of the class web page](#)

16. (expires 3/15) We saw that in order for an affine cipher given by $x \mapsto ax + b \pmod p$ to be invertible, we must be able to solve the equation $ax = 1 \pmod p$ for all nonzero values of x . This is equivalent to the condition $\gcd(a, p) = 1$, that is, that a and p must have no common divisors.

Something similar is true in the case when we do affine enciphering with a matrix (i.e. $v \mapsto Av + b \pmod p$, where A is an $n \times n$ matrix and b and v are n -vectors). What is the exact condition needed on A and p for mapping to be invertible, that is, for $A^{-1} \pmod p$ to exist? You don't have to give a proof of your answer (although you can), but use Maple to demonstrate several examples where the inverse exists and when it will fail to exist. (Obviously if the matrix A has a zero determinant, the inverse will not exist. But there are matrices which are invertible over the reals which do not have inverses $\pmod p$.)

17. (expires 3/15) Modify the `AffineMatCrypt` routine we wrote in class on [March 6](#) so that you can use a text string as a key instead of a matrix and a vector. (Alternatively, you can modify `AffineMatEncode` from section 8.2 of the notes, which is essentially the same thing.)

For example, if the phrase is k characters long, the key should be an $n \times n$ matrix and an n -vector, where $n^2 + n \approx k$, and n is determined from the length of the key phrase. The elements of the key matrix and vector should be the numerical equivalents of the characters in the key phrase. Do something sensible with any extra letters (that is, if $k \neq n^2 + n$). Be sure to check that the resulting matrix is nonsingular.

Demonstrate that your modifications work by encoding and decoding the text

```
Once upon a midnight dreary, while I pondered, weak and weary,
Over many a quaint and curious volume of forgotten lore,
While I nodded, nearly napping, suddenly there came a tapping,
As of some one gently rapping, rapping at my chamber door.
'Tis some visitor," I muttered, "tapping at my chamber door.
Only this, and nothing more."
```

with both a short key phrase (for example, *Lenore*, which would use a 2×2 matrix and a 2-vector) and with a longer keyphrase (for example, *Quoth the raven, "Nevermore"*, which requires either a 5×5 matrix and a 5-vector, or a 4×4 depending on how you treat extra letters.)

The text to encrypt can be found in the problems section of the class web page, in the file `raven.txt`.

18. (*expires 3/29*) Recall that a Vignère cipher can be interpreted as a Caesar-like cipher on n -vectors, where n is the length of the key phrase. Can every affine encipherment on digraphs (two-character codes) be interpreted as an affine matrix encipherment on 2-vectors? That is, suppose I encode a message by affine enciphering on digraphs. Can I always get the same ciphertext from the same plaintext using an affine matrix enciphering (using a 2×2 matrix) on 2-vectors? If your answer is yes, prove it. If no, give a counter-example that cannot be so interpreted.
19. (*expires 3/29*) Twenty-one pirates are dividing their horde of gold doubloons. Since they are a democratic outfit, they first try to divide the coins evenly, but they find there are 19 coins left over. The “discussion” about how to divide the remaining coins results in only 16 pirates still needing to divide the horde (the remaining five went to a place where you can’t bring money or anything else with you). The redivision among 16 pirates leaves 1 coin left over, and three of the pirates make a grab for it. These three find themselves to be missing their hands after this attempt, and the remaining thirteen pirates decide to divide the share among themselves, leaving the handless ones with nothing. Fortunately, the horde divides evenly among the thirteen. What is the minimum number of coins in the horde?
Hint: use the Chinese Remainder Theorem. The maple command `chrem` may be useful.
20. (*expires 3/29*) Use RSA with the modulus $n = 119$ and the exponent $e = 7$, with the 95-character alphabet consisting of the printable ASCII characters to encrypt the word “Yes”. Recall that the alphabet is given by
- ```
Alphabet:=cat(Select(IsPrintable,convert([seq(i,i=1..255)],bytes)));
```
- so that  $Y=57$ ,  $e=69$ ,  $s=83$ . Give your encryption as a list of three numbers.
21. (*expires 3/29*) With the same setup as the previous problem (that is  $n = 119$ ,  $e = 7$ ), the message after encrypting with RSA is the list of numbers

$$[51, 30, 23, 27, 27, 23, 70, 1].$$

Decrypt the message. (This is easily done because 119 is easy to factor).

22. (*expires 4/12*) Historically, “pen and paper” encryption (that is, what was mostly done prior to about 1980) would first rewrite a message to be encrypted in all upper-case letters, omitting all punctuation and spaces, and then writing out the message in blocks of five letters. For example, the message `Mr. Watson, come here - I want to see you!` would be transcribed as `MRWAT SONCO MEHER EIWAN TTOSE EYOU` prior to encrypting. If the message was sufficiently long (longer than 50 characters or so), the blocks of letters would be continued on a new line.

Write a pair of Maple procedures to accomplish this goal (in slightly more general terms).

The first should take as input a string, convert all lower-case letters to upper case, and afterwards, remove any characters that are not in a global variable called `Alphabet`. For this, I recommend using `UpperCase` and `Select` from the `StringTools` package.

The second procedure should print this out in a format as described above (blocks of 5 letters, separated by spaces). I recommend using `printf` for this.

As usual, demonstrate that your routines work with some appropriate test cases, with at least one short message as above, and another message of perhaps 100 characters or so. Note that your routine should work with a variety of alphabets.

23. (*expires 4/12*) When we implemented RSA in class, we represented our encrypted messages as a list of large numbers, rather than converting them to printable text. Sometimes we want a text representation. One way to do this is to use a base-64 representation, where the message  $m$  is converted to a base 64 number. This base 64 number is commonly represented with the upper-case characters A–Z representing digits 0 through 25, lower-case a–z representing digits 26 through 51, the characters 0–9 representing 52 through 61, and + and / representing 62 and 63, respectively.

Write a generalized implementation of this conversion process. Specifically, assume there is a global called `AlphabetOut` which contains the allowed characters in the encoding, ordered appropriately. Your procedure should take as input two arguments: a list of numbers in base  $n$ , and the base  $n$ . Your procedure should return a string representing the message in base  $b$ , where  $b$  is the length of `AlphabetOut`. Also write another procedure which undoes this conversion.

As an example<sup>1</sup>, the following `list of numbers` represents some text converted from ASCII (base 256) to base  $10^{47}$  (that is, without encryption):

```
6669013858395040150291124122141963456189571137,79085785195062207278272120198122975492318184082,
15624867350544934942834543866565863795868490456,55387683611779270304689525842891535523935500393,
23896957611465431133603100420167106476881540779,78091587231828327640146863695263953922927490912,
9764606424846784132731915166644883269708742761,24147181471923289394456210178528341598920602555,
59677642432524170063171614096094265077340147036,101708358765089971113312874866
```

When transcribed into the base 64 encoding described above, we get:

```
BBCbv52ZgQXatVGIhd2bsASauBSYgcWYsFGe5BiZhJHLgYWYyBSY3FWeu4iLKoQS
0BSazBSYgAXZyl2bkBybmByYpZXasBydhJnLgIVZiVGbgowcwF2YlNHapB3csAyc
0JXarlmbnBiZy9WbgEGIoIGZkVmbgogYhNXZsACahZXZgc3buBCdoVWayBiZpJ3c
0BidpNGdvJXegoQYnFWauNHdgQHalBSZ21GbgcUYsF2Y012YgUUbwlmc15C
```

By the way, this text is from the opening to a well-known movie.

<sup>1</sup>Using Maple’s convention of least-significant digit first, so the decimal number 123 is [59,1] (or 7B) in base 64.

24. (*expires 4/12*) The difficulty of breaking a cipher can be increased by inserting some random characters (or noise) into a known part of the plain text in such a way that it will interact with the actual text (sometimes this insertion of randomness is called “salting the plaintext”).

As an example of this, recall that we discussed that if our character set came from an alphabet of length  $n$ , we could represent blocks of  $k$ -characters as numbers base  $n^k$  (for example, in the 53-character alphabet consisting of a space, upper-case letters and lower-case letters, the word `Hi` is represented by  $8 + 35 \cdot 53 = 1863$  if we use 2-character blocks).

If, instead, we agree up front that the first character of each block will be randomly chosen (and just ignored when we decrypt), breaking the encryption becomes much, much harder.

Modify the affine encryption scheme we discussed in class to encrypt and decrypt with character blocks of any length, including salt as the first character of each block.

As an explicit example, we encode the string `Zombie Apocalypse` using the 53-character alphabet and 3-character blocks (where the first character of each block is random noise). The given string corresponds to the list of character codes

$$[26, 41, 39, 28, 35, 31, 0, 1, 42, 41, 29, 27, 38, 51, 42, 45, 31]$$

which, when grouped into pairs and with a random character added as salt, gives me

$$[116551, 80721, 88936, 2842, 117404, 77428, 145275, 128676, 1652]$$

(your numbers may differ by up to 53 because of the salt); viewing these as 3-grafs including the salt, corresponds to `DZoBmbBieg AIpovcaBlyspIe`. If we encrypt this using the affine encryption

$$x \mapsto 12347x + 56890 \pmod{53^3},$$

we get

$$[67005, 136439, 32930, 12092, 28729, 121189, 97219, 4118, 57985]$$

or `MsWQdvQlKHPDCLJeGqQfhkXACHT` (unless you used the same salt, your encryption will differ significantly, but both should decrypt just fine.)

If you were able to make this work, you should be able to decrypt the string<sup>2</sup>

`MWLiuWVckaMOpfHuWPgGuWlyQovqkwBwWV nLZYhrySYihTUSFqFJuGxEtvxCWNxPxstkPwkkxo`

which was encrypted using salted pairs (that is, 3-character blocks including salt) from the 53-character alphabet above, and applying the affine mapping  $x \mapsto 47x + 31415 \pmod{53^3}$ .

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<sup>2</sup>also available from [the class web page](#) so you don't have to worry about typing errors.

25. (expires 4/12) The message below was encrypted using RSA with a key having the base

$$n = 27606985387162255149739023449107931668458716142620601169954803000803329$$

and the exponent

$$e = 95541407564551142884433930859007846184688349517132011538195190379009$$

The plaintext was represented in printable ASCII (base 128) in blocks of 32 letters at a time, and no padding.

[763135969033211128596541748452730974762878358020265079326172641096139,  
 20879815087895860022209511826785258055291427630017621830659043784235896,  
 5574407969259647015040456167101346596832809056832375987068556624523356,  
 21224197154621559219944504912186017011922165030059099210273090531937808,  
 12551200560613077704270621749705141076270903046687425935996313400405711,  
 18867900358254708499609170144443800027105466906756657749060030225245438,  
 4496337993898968047719657088708600524228317595964529102107251777431690,  
 9235737130358789283717978176700453123390494678379655510638922958288836,  
 3231140561089428768657095292397775917600836123018152094745253390870101,  
 82840704708021038125667422565345512548754665959685316382831252550171,  
 21039656037112093852283327710959368295150841772122729298435513871462186,  
 12041411940954720052070811665940863639535876777498117250838013124490438,  
 9413438705357919431318689662458482172376428983918258919377408820020874,  
 24344582781105714296470393649838967454596532389718150304278089340495219,  
 11666627210237277694133981546667501319351377137704324464898696799000305,  
 7678728878370382868719828867510796171125030253163810615836093095505785,  
 5850521474451619072915665399042845983658256530488470900878348493299278,  
 12796063700367369818580350073200291698274548230387522514837595194567646,  
 991710678121376154332316965121228090363454988579278882664135678580849,  
 5448263916134504769205437219736103649161232816051564930186689344069733,  
 331683831213526862116612352660410874927161591545716084674509520546695,  
 20911409982862166039437808848985365838241145413477157865204152349571315,  
 8446725165717024647474699066623651985526914825302054942325942578083958,  
 14694800659219414119256346254478510304289471490144936213050026188930109,  
 7890485067636181638866077827274071592238401748222991089801197812460720,  
 21236486045506721843702186793969054222520252187939247582471851381880906,  
 26957526821518528986780575261941374306276897675892965905302075045418371,  
 24677240511277808315835169523848692305481480201233529864615262657333913,  
 23499633482301313755682672041801597278928819855701032620066320680181087,  
 372227370145476230598908963364257918250207931059342487415859925223516]

Decrypt the message. (Hint: 70 decimal digits is too small to be secure.) So you don't have to retype everything, the values of  $n$ ,  $e$ , and the encrypted message are in the file [formulacrypt.txt](#) on the class web page.

26. (expires 4/19) Find a formula for all the solutions to the differential equation

$$\frac{dx}{dt}(t) = -2x(t), \quad t \in \mathbb{R}.$$

Among them, single out the one for which  $x(0) = 3$ . [Hint: read the help page for `dsolve`, or just do it in your head. It is that easy.]

27. (expires 4/19) Have Maple find analytic solutions to the following system of differential equations,

$$\begin{cases} y''(t) - z(t) = e^t, \\ z'(t) - y(t) = 0, \end{cases}$$

with initial conditions:  $y(0) = 1$ ,  $y'(0) = 0$ ,  $z(0) = k$ . Let us denote the solutions by  $y_k(t)$ ,  $z_k(t)$  (since they depend on the parameter  $k$ ).

For  $k$  taking all integer values from -10 to 10, and  $t \in [-4, 2]$ , plot the functions  $y_k$  in blue, and the functions  $z_k$  in red, all on the same graph. (Yes, you will then have 42 functions plotted on the same graph.) [This is certainly a case when you don't want to retype the functions that Maple finds. You will almost certainly need to read the help page for `dsolve`. I also found `subs`, `unapply`, and `seq` useful.]

28. (expires 4/19) For the functions  $y_k(t)$  and  $z_k(t)$  found in the previous problem, plot the parametric curves  $\varphi_k(t) = [y_k(t), z_k(t)]$  for integer values of  $k$  between  $-5$  and  $5$  and  $-6 < t < 4$  on the same graph. Use the `view` option of `plot` to only show what lies in the region  $-10 < y < 10$ ,  $-10 < z < 10$ , and use a sequence of colors so that each solution is a different color. [HINT: you might find something like `seq(COLOR(HUE, i/11), i=0..10)` useful for the latter.]

29. (expires 4/19) Find all the fixed points of the system

$$\begin{cases} \dot{x} = x^2 + y, \\ \dot{y} = x(y^2 - 1), \end{cases}$$

where a "fixed point" is a solution for which **both**  $x(t)$  **and**  $y(t)$  are constant. For each of these solutions you find, describe the behavior of the solutions that have initial conditions nearby. You can use Maple to figure out what happens for nearby points, or you can use more mathematical methods.

**NOTE:** The fact that there are various notations for differential equations is purely intentional.

30. (expires 5/3) Consider the differential equation  $\dot{\mathbf{z}}(t) = \mathbf{F}(\mathbf{z}(t))$ , where the vector  $\mathbf{z}(t) = (x(t), y(t))$  and the field  $\mathbf{F}(x, y) = (-y, x - y)$ . Plot a few solutions. What happens to them when  $t \rightarrow +\infty$ ? Give a “Maple-proof” that this is a general fact for *every* solution. [A “Maple-proof” is an argument that is rigorous once we accept Maple results as incontrovertibly true.]
31. (expires 5/3) For the equation  $\dot{\mathbf{z}} = \mathbf{F}(\mathbf{z})$ ,  $\mathbf{z} = (x, y)$ , with the vector field

$$\mathbf{F}(x, y) = \langle -x(x^4 + y^4) - y, x - y(x^4 + y^4) \rangle,$$

prove that the origin is an attractor in the future, i.e., every solution verifies

$$\lim_{t \rightarrow +\infty} \mathbf{z}(t) = 0.$$

Note that in this case, while you can (and probably should) use Maple to get an understanding and do calculations, you should format your answer as a regular mathematical proof.

32. (expires 5/3) We will study the Lotke-Volterra predator-prey equations: In a very simple ecosystem, at the time  $t$  (which is expressed, say, in years), there is a population of  $f(t)$  foxes and  $r(t)$  rabbits. The evolution of these quantities obeys the system

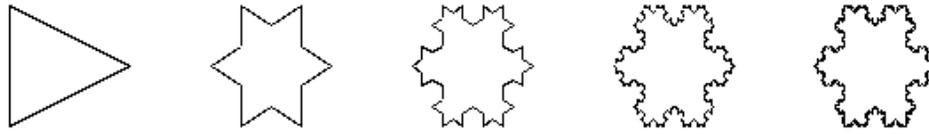
$$\begin{cases} \dot{f}(t) = G_f f(t) + E f(t) r(t), \\ \dot{r}(t) = G_r r(t) - E f(t) r(t); \end{cases}$$

where  $G_f$  and  $G_r$  are the growth rates for the foxes and the rabbits, respectively, in the absence of each other.  $E$  is the probability of a fatal encounter between a fox and a rabbit (normalized per number of foxes and rabbits).

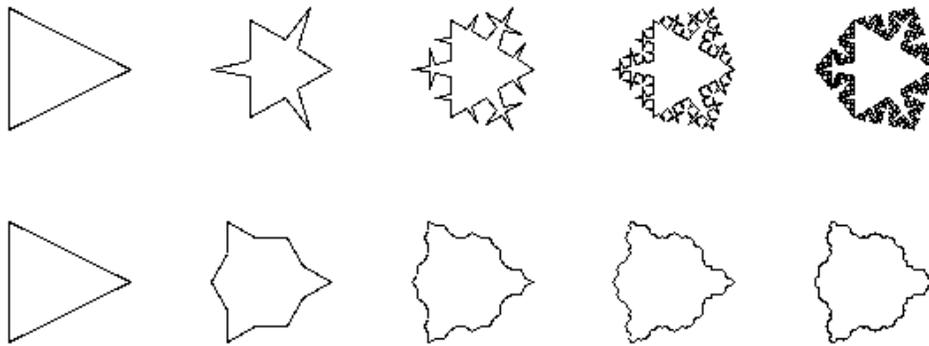
First, write some words to explain why these equations make sense. Then, fix  $G_f = 0.4$ ,  $G_r = 2.4$  (it’s well-known that rabbits have the tendency to reproduce quickly) and  $E = 0.01$ . For a few initial conditions of your choice, plot the trajectories in the  $(f, r)$ -plane (say, with  $0 \leq f \leq 1000$  and  $0 \leq r \leq 1000$ ). For the same initial conditions, plot the actual solutions too (i.e,  $f(t)$  against  $t$ , and  $r(t)$  against  $t$ ). Write some comments interpreting how the behaviour of the solutions relates to what happens to the two species.

Finally, repeat the same procedure with  $G_f = -1.1$ . Things change substantially. Again, what is the “physical” interpretation of this?

33. (*expires 5/10*) Using the turtle commands as discussed in class, create the vonKoch snowflake by modifying what we did in class to put three copies of the vonKoch curve together, as below.



34. (*expires 5/10*) If you did the above problem, modify it to make a “spikier” and “flatter” version of the snowflake, similar to those below.



Given the choices you made above, calculate the fractal dimension of the limiting figures, as discussed in the class notes (but not really in class). (Hint: it is easier to do the calculations for some choices of angles than others.)

35. (*expires 5/10*) Modify the example of making the treelike shape done in class to make a fractal that looks like a fern.
36. (*expires 5/10*) Use the turtle software to make the fractal whose successive refinements are as below

