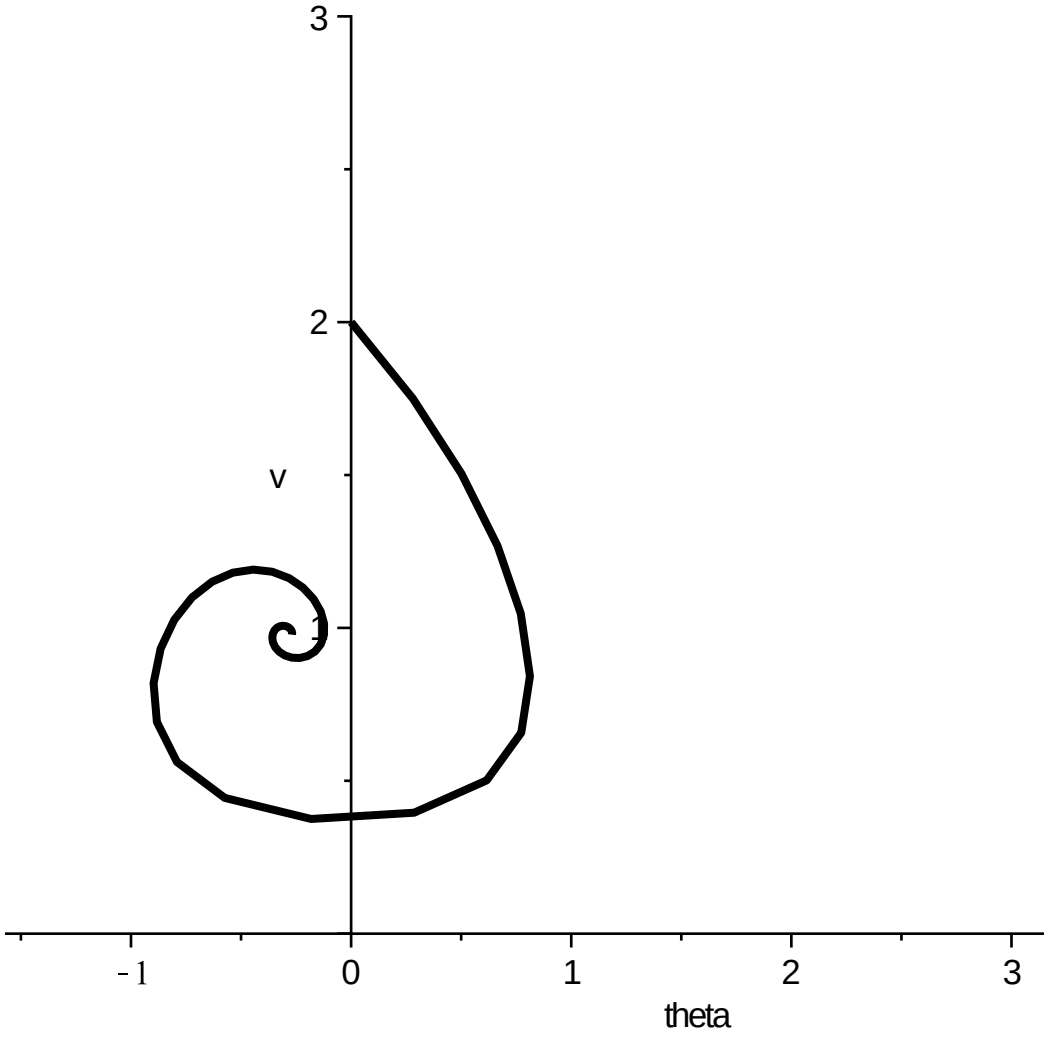
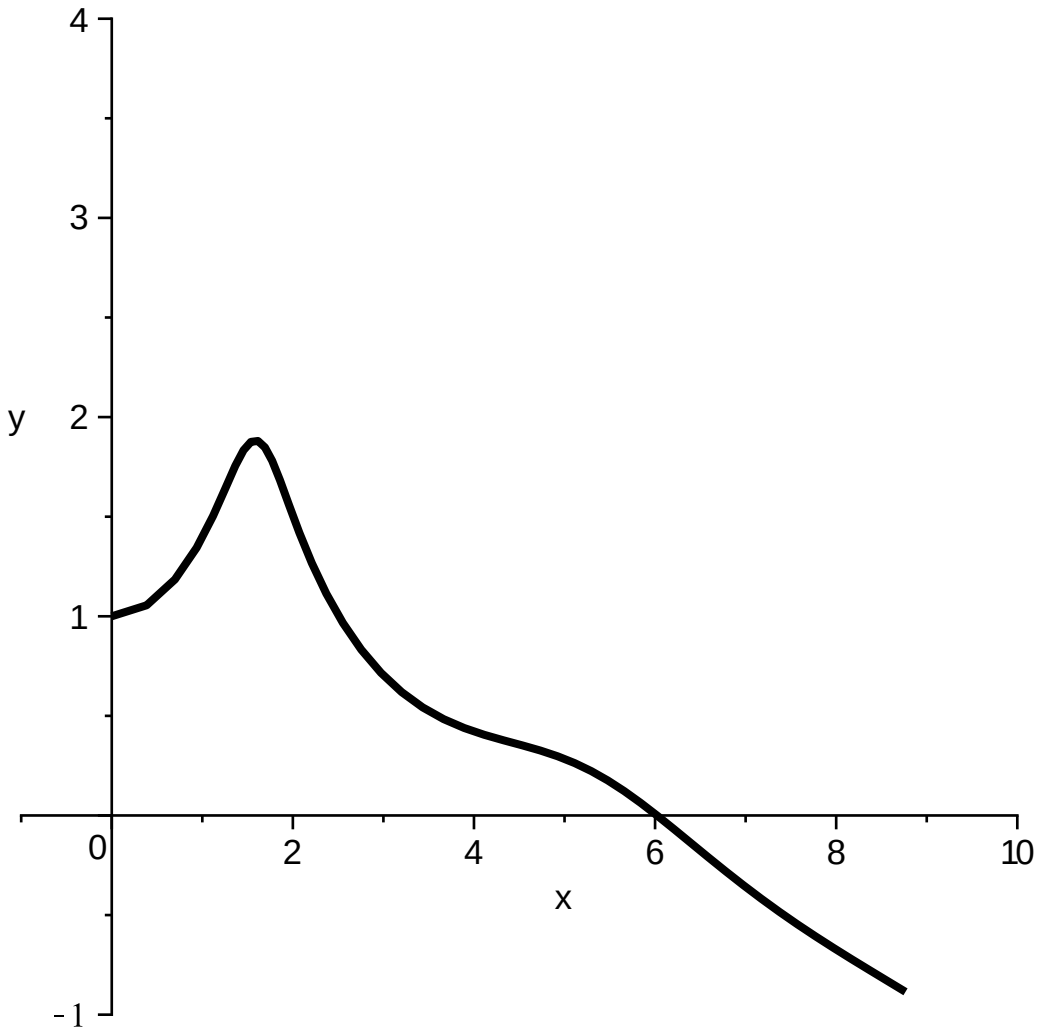
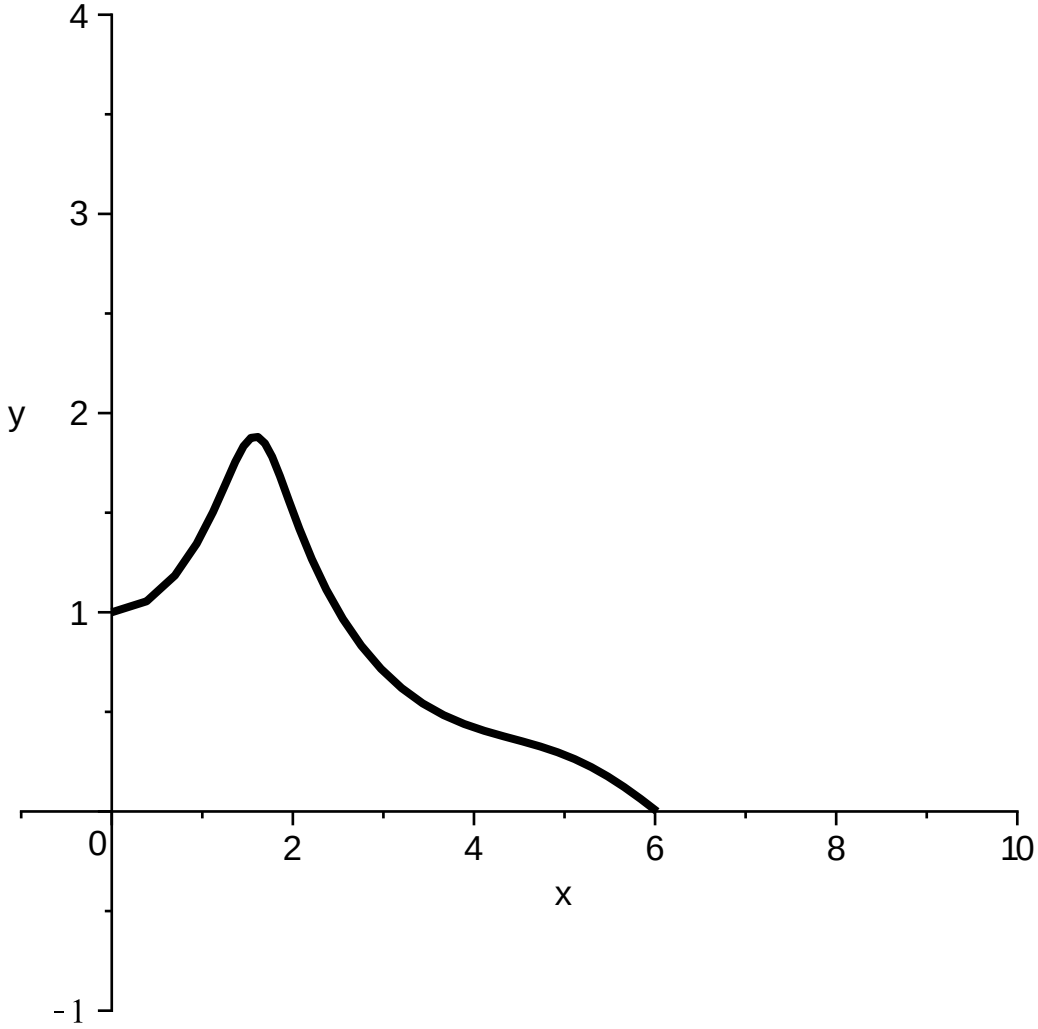
```
> R:=0.3:
> xphug:= [ diff(theta(t),t) = ( v(t)^2 - cos(theta(t))) / v(t),
>        diff(v(t),t)     = -sin(theta(t)) - R*v(t)^2 ,
>        diff(x(t),t)     = v(t)*cos(theta(t)),
>        diff(y(t),t)     = v(t)*sin(theta(t))]:

> with(DEtools): with(plots):
> DEplot( xphug, [theta(t), v(t),x(t), y(t)], t=0..10,
     theta=-Pi/2..Pi, v=0..3, x=-1..4, y=-1..4,
    [[theta(0)=0, v(0)=2, x(0)=0, y(0)=1]],
    scene=[theta,v], linecolor=black);
```




```
> DEplot( xphug, [theta(t), v(t),x(t), y(t)], t=0..10,
     theta=-Pi/2..Pi, v=0..3, x=-1..10, y=-1..4,
    [[theta(0)=0, v(0)=2, x(0)=0, y(0)=1]],
    scene=[x,y], linecolor=black);
```

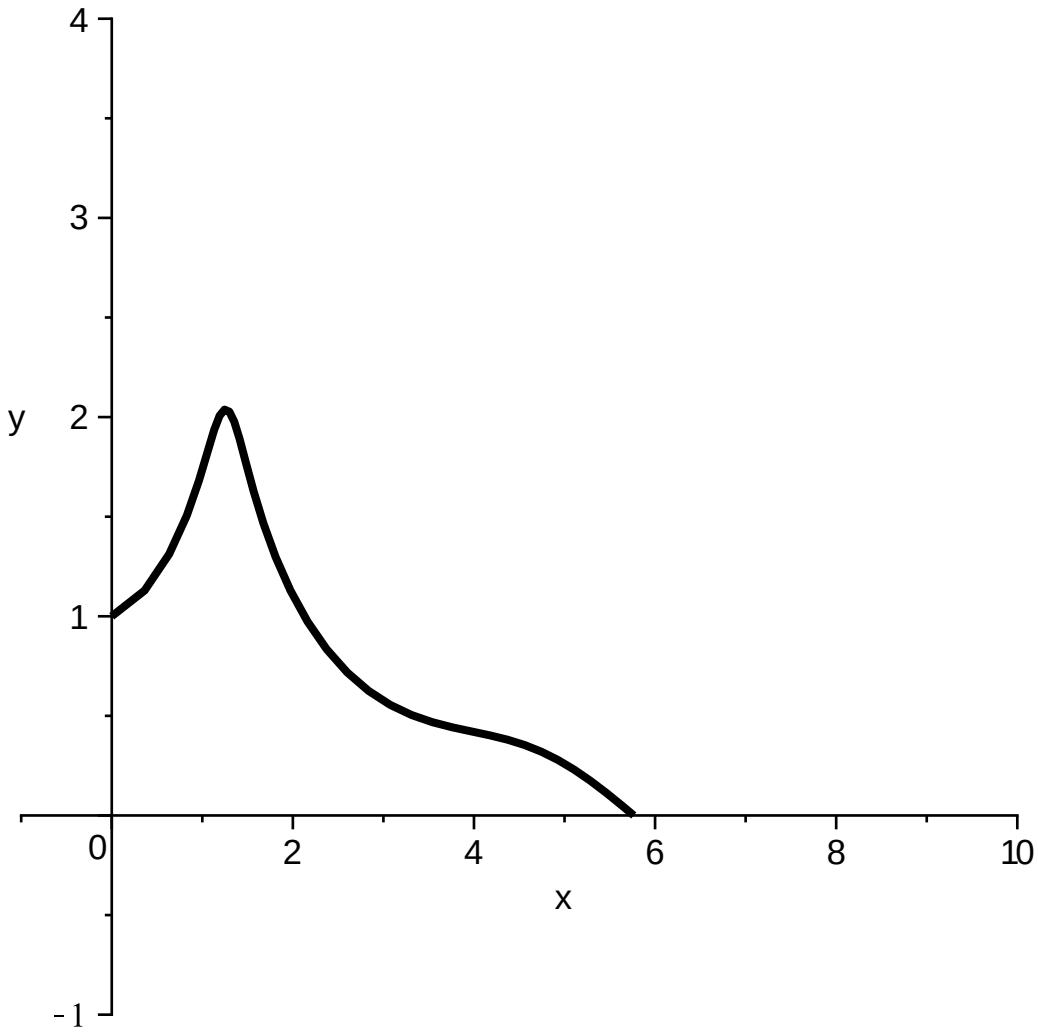



```
> crash:=[ D(theta)(t) = piecewise( y(t)>0,v(t) - cos(theta(t))/v
  (t),0),
        D(v)(t)    = piecewise( y(t)>0,-sin(theta(t)) -R*v(t)^2,
  0),
        D(x)(t)    = piecewise( y(t)>0,v(t)*cos(theta(t)),0),
        D(y)(t)    = piecewise( y(t)>0, v(t)*sin(theta(t)),0)]:
> DEplot( crash, [theta(t), v(t),x(t), y(t)], t=0..10,
     theta=-Pi/2..Pi, v=0..3, x=-1..10, y=-1..4,
    [[theta(0)=0, v(0)=2, x(0)=0, y(0)=1]],
    scene=[x,y], linecolor=black);
```

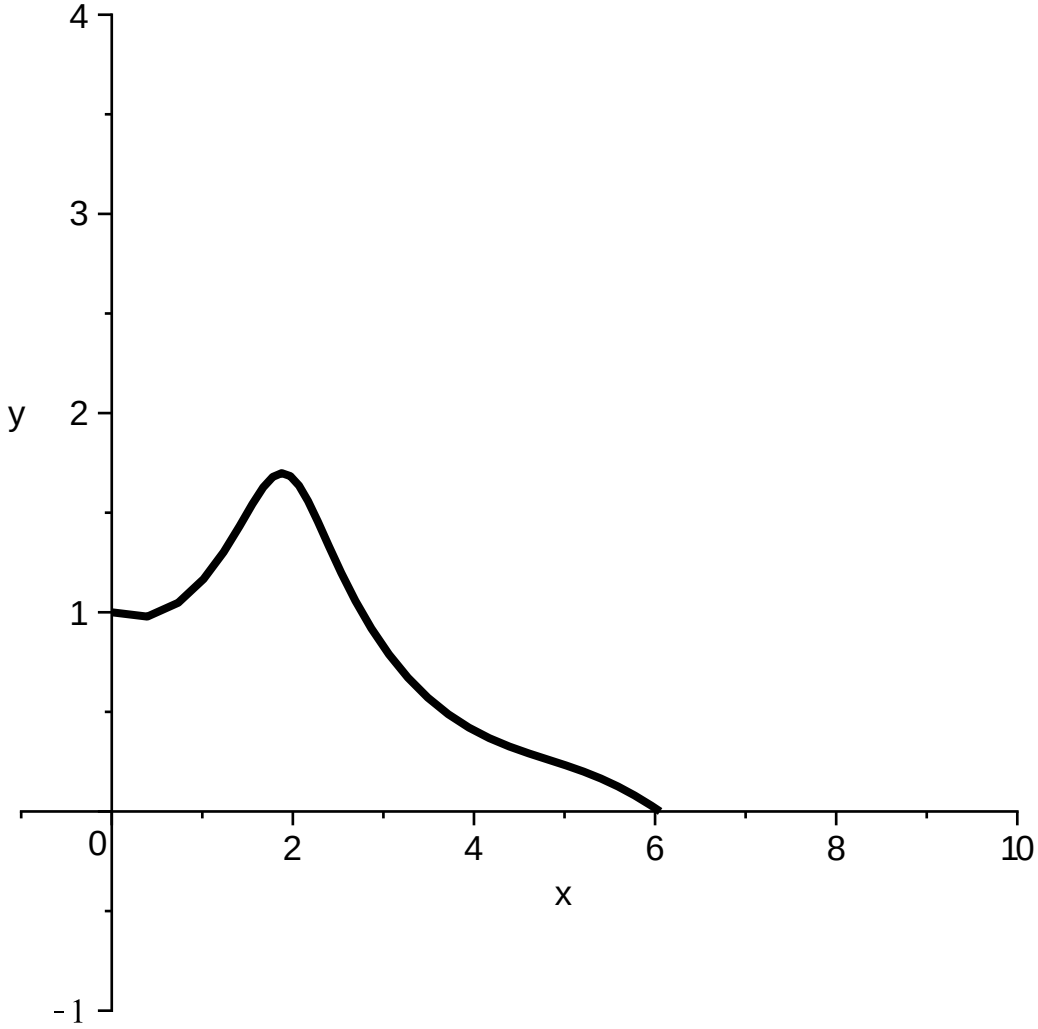


Increasing the angle goes less far.

```
> DEplot( crash, [theta(t), v(t),x(t), y(t)], t=0..10,
    theta=-Pi/2..Pi, v=0..3, x=-1..10, y=-1..4,
   [[theta(0)=0.2, v(0)=2, x(0)=0, y(0)=1]],
   scene=[x,y], linecolor=black);
```




```
> DEplot( crash, [theta(t), v(t),x(t), y(t)], t=0..10,
    theta=-Pi/2..Pi, v=0..3, x=-1..10, y=-1..4,
   [[theta(0)=-0.2, v(0)=2, x(0)=0, y(0)=1]],
   scene=[x,y], linecolor=black);
```




```
> crash;
```

$$\left[\mathrm{D}(\theta)(t) = \begin{cases} v(t) - \dfrac{\cos(\theta(t))}{v(t)} & 0 < y(t) \\ 0 & otherwise \end{cases}, \mathrm{D}(v)(t) = \right. \tag{1}$$

$$\begin{cases} -\sin(\theta(t)) - 0.3\, v(t)^2 & 0 < y(t) \\ 0 & otherwise \end{cases}, \mathrm{D}(x)(t) = \begin{cases} v(t)\cos(\theta(t)) & 0 < y(t) \\ 0 & otherwise \end{cases},$$

$$\left. \mathrm{D}(y)(t) = \begin{cases} v(t)\sin(\theta(t)) & 0 < y(t) \\ 0 & otherwise \end{cases} \right]$$

```
> sol2:=dsolve( {op(crash), theta(0)=-0.2, v(0)=2, x(0)=0, y(0)=1},
  numeric);
```

$$sol2 := \mathbf{proc}(x\_rkf45)\ \ldots\ \mathbf{end\ proc} \tag{2}$$

```
> sol2(1);
```

$$[t = 1.,\ \theta(t) = 0.655873698062745,\ v(t) = 0.990844860245643,\ x(t) = 1.37942187550901,\ y(t) \tag{3}$$

$= 1.40745905942525\ ]$

```
> sol2(7);
```

$$[t = 7.,\ \theta(t) = -0.299549827167121,\ v(t) = 0.919451393794940,\ x(t) = 6.05552404304945,\ y(t) = -3.59573691168880\ 10^{-8}] \quad (4)$$

Initial condition theta=-0.2 goes 6.05... units in x.

```
> sol3:=dsolve( {op(crash), theta(0)=-0.3, v(0)=2, x(0)=0, y(0)=1},
  numeric);
```

$$sol3 := \mathbf{proc}(x\_rkf45)\ \ldots\ \mathbf{end\ proc} \quad (5)$$

```
> sol3(7);
```

$$[t = 7.,\ \theta(t) = -0.260982372247784,\ v(t) = 0.916239115288705,\ x(t) = 5.97035557175574,\ y(t) = -1.26448066093737\ 10^{-7}] \quad (6)$$

```
> myx:= subs(dsolve( {op(crash), theta(0)=-0.3, v(0)=2, x(0)=0, y
  (0)=1}, numeric, output=listprocedure), x(t));;
```

$$myx := \mathbf{proc}(t)\ \ldots\ \mathbf{end\ proc} \quad (7)$$

```
> myx(5);
```

$$4.50239678786362 \quad (8)$$

```
> myx(7);
```

$$5.97035557175574 \quad (9)$$

```
> dist:= theta0 -> dsolve( {op(crash), theta(0)=theta0, v(0)=2, x
  (0)=0, y(0)=1}, numeric, output=listprocedure);
```

$$dist := \theta 0 \rightarrow dsolve(\{op(crash), v(0) = 2, x(0) = 0, y(0) = 1, \theta(0) = \theta 0\}, numeric, output = listprocedure) \quad (10)$$

```
> dist(0);
```

$$[t = \mathbf{proc}(t)\ \ldots\ \mathbf{end\ proc},\ \theta(t) = \mathbf{proc}(t)\ \ldots\ \mathbf{end\ proc},\ v(t) = \mathbf{proc}(t)\ \ldots\ \mathbf{end\ proc},\ x(t) = \mathbf{proc}(t)\ \ldots\ \mathbf{end\ proc},\ y(t) = \mathbf{proc}(t)\ \ldots\ \mathbf{end\ proc}] \quad (11)$$

```
> dist(0)[4](10);
```

$$x(t)(10) = 6.02160894819366 \quad (12)$$

```
> dist(-0.3)[4](10);
```

$$x(t)(10) = 5.97035557175574 \quad (13)$$

```
> f:= theta->dist(theta)[4](10);
```

$$f := \theta \rightarrow dist(\theta)_4(10) \quad (14)$$

```
> f(-0.3);
```

$$x(t)(10) = 5.97035557175574 \quad (15)$$

```
> plot(f(theta), theta=-0.5..0.5);
```

Warning, The use of global variables in numerical ODE problems is deprecated, and will be removed in a future release. Use the 'parameters' argument instead (see ?dsolve,numeric,parameters)

Error, (in dsolve/numeric) theta(t) and theta cannot both appear in the given ODE

```
> rhs(dist(-0.3)[4](10));
```

$$5.97035557175574 \quad (16)$$

```
> f:=theta->rhs(dist(theta)[4](10));
```

$$f := \theta \rightarrow rhs\left(dist(\theta)_4(10)\right) \quad (17)$$

```
> f(-0.3);
```

5.97035557175574 (18)

```
> f(0);
```

6.02160894819366 (19)

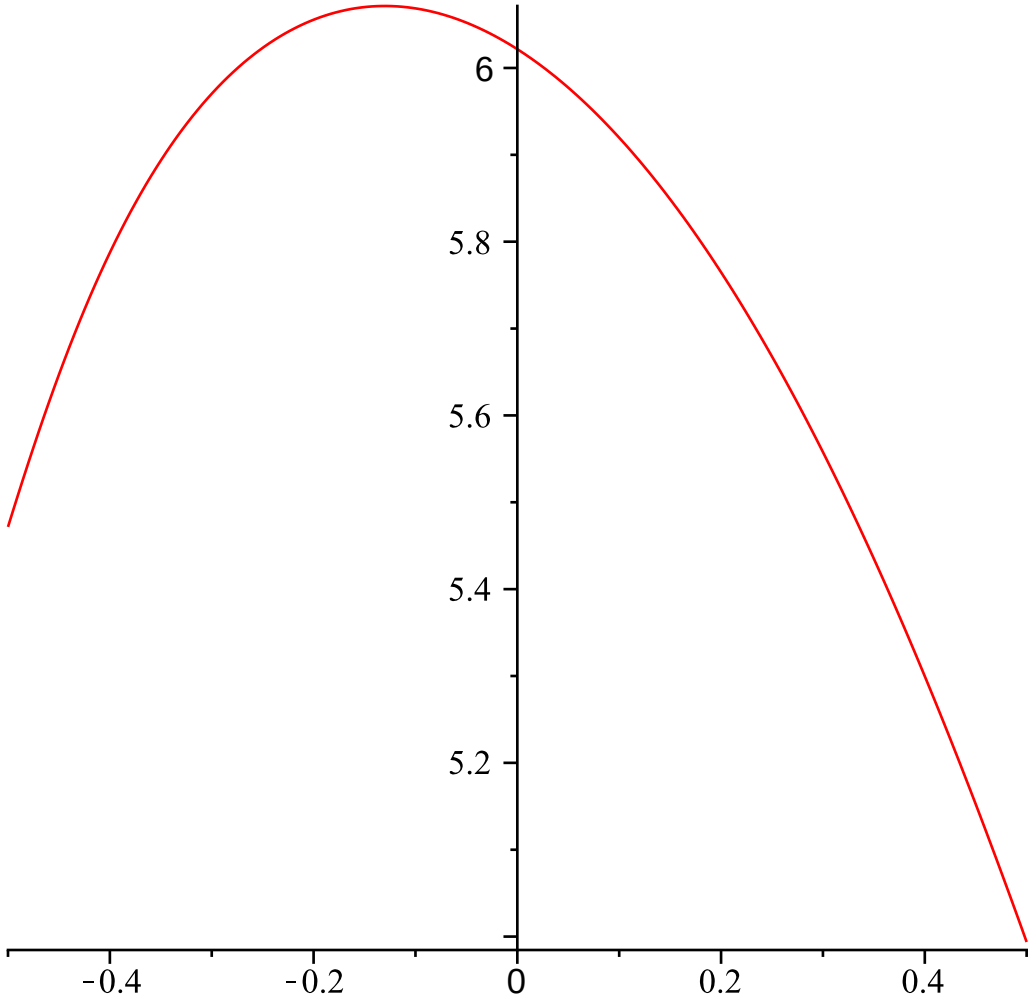
```
> f(1);
```

3.15678789024954 (20)

```
> plot(f(walrus),walrus=-0.5 .. 0.5);
```

Warning, The use of global variables in numerical ODE problems is deprecated, and will be removed in a future release. Use the 'parameters' argument instead (see ?dsolve,numeric,parameters)
Error, (in unknown) parameter 'walrus' must be assigned a numeric value before obtaining a solution

```
> plot(f,-0.5 .. 0.5);
```

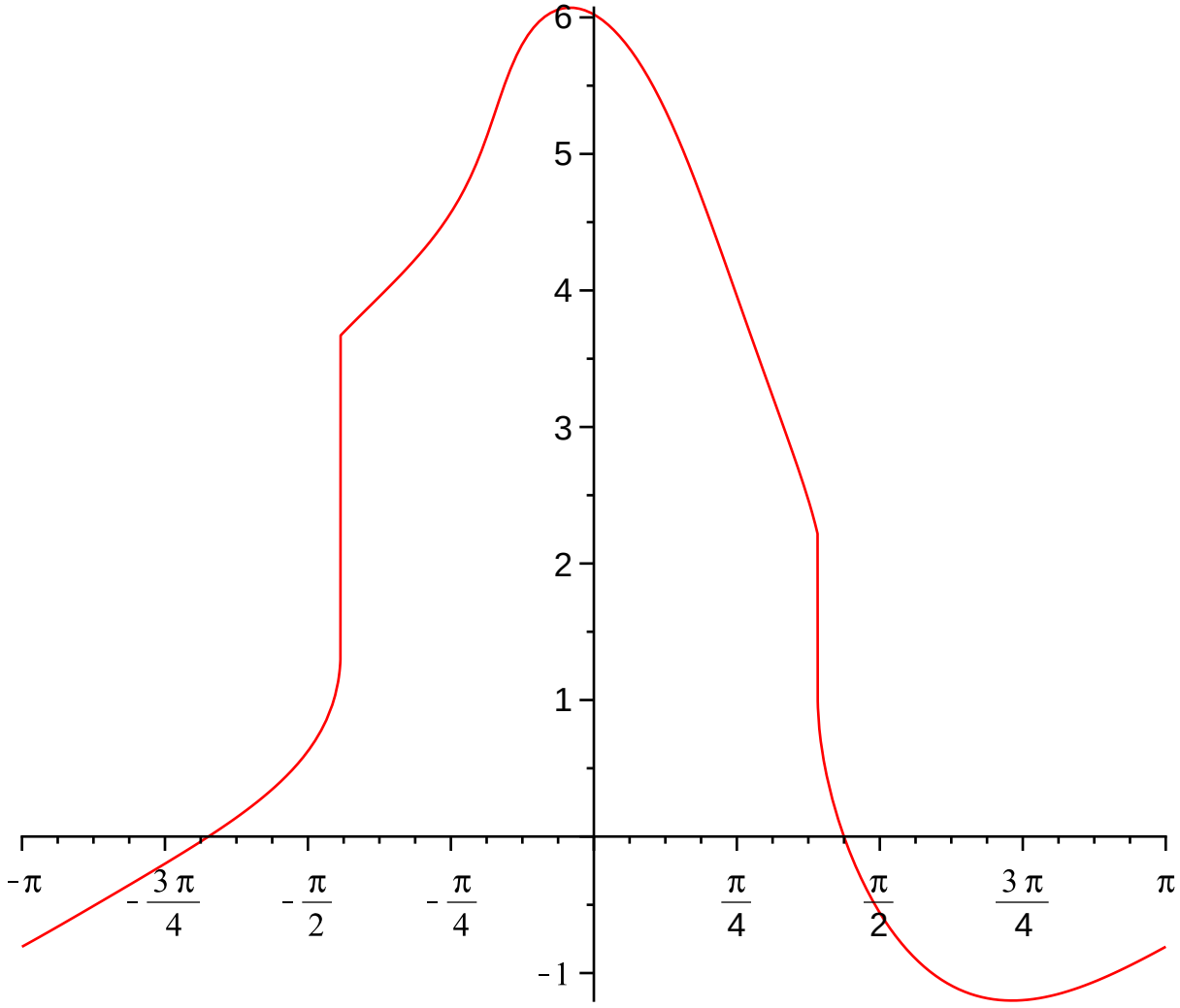



```
> f(walrus);
```

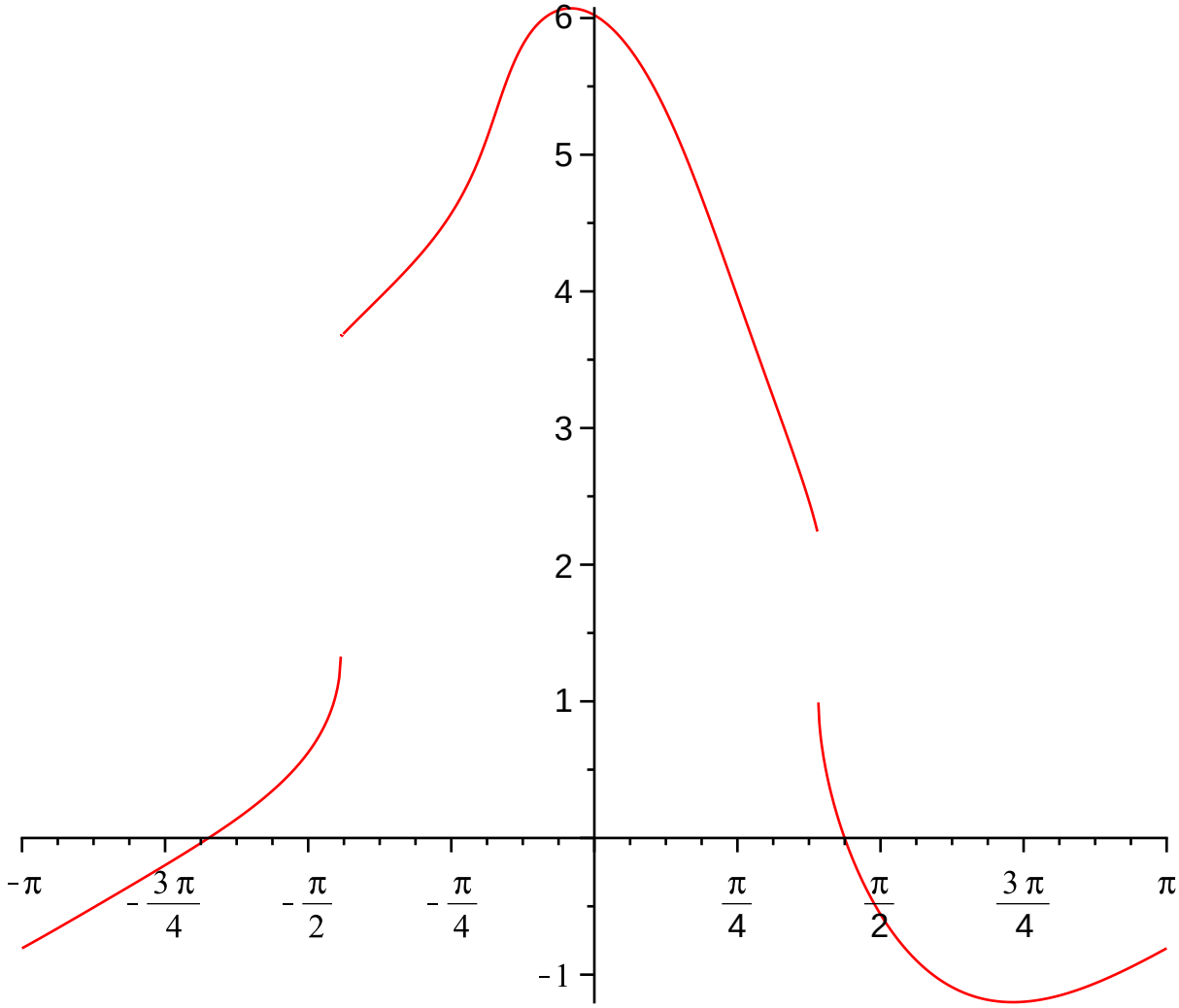
Warning, The use of global variables in numerical ODE problems is deprecated, and will be removed in a future release. Use the 'parameters' argument instead (see ?dsolve,numeric,parameters)
Error, (in unknown) parameter 'walrus' must be assigned a

numeric value before obtaining a solution

```
> plot(f,-Pi .. Pi);
```




```
> plot(f,-Pi .. Pi, discont=true);
```




```
> dsolve( {op(crash), theta(0)=0.57, v(0)=2, x(0)=0, y(0)=1},
  numeric)(10);
```

$$[t = 10.,\ \theta(t) = 5.88977819525915,\ v(t) = 0.944988975829048,\ x(t) = 4.75542620786890,\ y(t) = -1.42577518797824\ 10^{-7}] \quad (21)$$

Why the discontinuity???? How would we know what is going on here?

you think about it.... different question: what initial angle goes 5 units?

```
> f(0.498);
```

5.00031414847021 (22)

```
> f(0.497);
```

5.00358250716599 (23)

```
> f(0.499);
```

4.99704188554361 (24)

```
> solve(f(x)=5, x);
```

Warning, The use of global variables in numerical ODE problems is deprecated, and will be removed in a future release. Use the

```
'parameters' argument instead (see ?dsolve,numeric,parameters)
Error, (in dsolve/numeric) x(t) and x cannot both appear in the
given ODE
> f(0.4985);
```

4.99867837488214 (25)

```
> f(0.49825);
```

4.99949651042563 (26)

```
> f(0.498175);
```

4.99974167290558 (27)

```
> f(0.498085);
```

5.00003641872253 (28)

```
> bisection:= proc( f, y, xhi, xlo, eps)
    local xmid;
    xmid:=(xhi+xlo)/2;

    while  (abs(f(xmid)-y) > eps) do
       if (f(xmid)> y) then
           xhi:=xmid;
        else
           xlo:=xmid;
         fi;
    end do;
    return(xmid);
  end;
```

$$bisection := \mathbf{proc}(f, y, xhi, xlo, eps) \quad (29)$$

```
    local xmid;
    xmid := 1 / 2 * xhi + 1 / 2 * xlo;
    while eps < abs( f (xmid) − y) do
        if y < f (xmid) then xhi := xmid else xlo := xmid end if
    end do;
    return xmid
end proc
> g:=x->x^2;
```

$$g := x \rightarrow x^2 \quad (30)$$

```
> bisection(g, 2, 0, 2, 0.001);
Error, (in bisection) illegal use of a formal parameter
> bisection:= proc( f, y, xh, xl, eps)
    local xmid, xhi, xlo, n;
    xhi:=xh;
    xlo:=xl;
    xmid:=(xhi+xlo)/2;
    n:=0;
    while (n<100 and (abs(f(xmid)-y) > eps)) do
       print(xlo,xmid,xhi,f(xmid));
       if (f(xmid)> y) then
           xhi:=xmid;
        else
           xlo:=xmid;
         fi;
```

```
      xmid:=(xhi+xlo)/2;
      n:=n+1;
    end do;
    return(xmid);
  end:
```

```
> bisection(g, 2, 2, 0, 0.001);
```

$$0, 1, 2, 1$$

$$1, \frac{3}{2}, 2, \frac{9}{4}$$

$$1, \frac{5}{4}, \frac{3}{2}, \frac{25}{16}$$

$$\frac{5}{4}, \frac{11}{8}, \frac{3}{2}, \frac{121}{64}$$

$$\frac{11}{8}, \frac{23}{16}, \frac{3}{2}, \frac{529}{256}$$

$$\frac{11}{8}, \frac{45}{32}, \frac{23}{16}, \frac{2025}{1024}$$

$$\frac{45}{32}, \frac{91}{64}, \frac{23}{16}, \frac{8281}{4096}$$

$$\frac{181}{128} \qquad (31)$$

```
> evalf(%);
```

$$1.414062500 \qquad (32)$$

Note that the following is a problem:

```
> bisection(g, 2, 0, 2, 0.001);
```

$$2, 1, 0, 1$$

$$1, \frac{1}{2}, 0, \frac{1}{4}$$

$$\frac{1}{2}, \frac{1}{4}, 0, \frac{1}{16}$$

$$\frac{1}{4}, \frac{1}{8}, 0, \frac{1}{64}$$

$$\frac{1}{8}, \frac{1}{16}, 0, \frac{1}{256}$$

$$\frac{1}{16}, \frac{1}{32}, 0, \frac{1}{1024}$$

$$\frac{1}{32}, \frac{1}{64}, 0, \frac{1}{4096}$$

$$\frac{1}{64}, \frac{1}{128}, 0, \frac{1}{16384}$$

$$\frac{1}{128}, \frac{1}{256}, 0, \frac{1}{65536}$$

$$\frac{1}{256}, \frac{1}{512}, 0, \frac{1}{262144}$$

$$\frac{1}{512}, \frac{1}{1024}, 0, \frac{1}{1048576}$$
$$\frac{1}{1024}, \frac{1}{2048}, 0, \frac{1}{4194304}$$
$$\frac{1}{2048}, \frac{1}{4096}, 0, \frac{1}{16777216}$$
$$\frac{1}{4096}, \frac{1}{8192}, 0, \frac{1}{67108864}$$
$$\frac{1}{8192}, \frac{1}{16384}, 0, \frac{1}{268435456}$$
$$\frac{1}{16384}, \frac{1}{32768}, 0, \frac{1}{1073741824}$$
$$\frac{1}{32768}, \frac{1}{65536}, 0, \frac{1}{4294967296}$$
$$\frac{1}{65536}, \frac{1}{131072}, 0, \frac{1}{17179869184}$$
$$\frac{1}{131072}, \frac{1}{262144}, 0, \frac{1}{68719476736}$$
$$\frac{1}{262144}, \frac{1}{524288}, 0, \frac{1}{274877906944}$$
$$\frac{1}{524288}, \frac{1}{1048576}, 0, \frac{1}{1099511627776}$$
$$\frac{1}{1048576}, \frac{1}{2097152}, 0, \frac{1}{4398046511104}$$
$$\frac{1}{2097152}, \frac{1}{4194304}, 0, \frac{1}{17592186044416}$$
$$\frac{1}{4194304}, \frac{1}{8388608}, 0, \frac{1}{70368744177664}$$
$$\frac{1}{8388608}, \frac{1}{16777216}, 0, \frac{1}{281474976710656}$$
$$\frac{1}{16777216}, \frac{1}{33554432}, 0, \frac{1}{1125899906842624}$$
$$\frac{1}{33554432}, \frac{1}{67108864}, 0, \frac{1}{4503599627370496}$$
$$\frac{1}{67108864}, \frac{1}{134217728}, 0, \frac{1}{18014398509481984}$$
$$\frac{1}{134217728}, \frac{1}{268435456}, 0, \frac{1}{72057594037927936}$$
$$\frac{1}{268435456}, \frac{1}{536870912}, 0, \frac{1}{288230376151711744}$$
$$\frac{1}{536870912}, \frac{1}{1073741824}, 0, \frac{1}{1152921504606846976}$$
$$\frac{1}{1073741824}, \frac{1}{2147483648}, 0, \frac{1}{4611686018427387904}$$

$$\frac{1}{2147483648}, \frac{1}{4294967296}, 0, \frac{1}{18446744073709551616}$$
$$\frac{1}{4294967296}, \frac{1}{8589934592}, 0, \frac{1}{73786976294838206464}$$
$$\frac{1}{8589934592}, \frac{1}{17179869184}, 0, \frac{1}{295147905179352825856}$$
$$\frac{1}{17179869184}, \frac{1}{34359738368}, 0, \frac{1}{118059162071741130342 4}$$
$$\frac{1}{34359738368}, \frac{1}{68719476736}, 0, \frac{1}{472236648286964521369 6}$$
$$\frac{1}{68719476736}, \frac{1}{137438953472}, 0, \frac{1}{188894659314785808547 84}$$
$$\frac{1}{137438953472}, \frac{1}{274877906944}, 0, \frac{1}{755578637259143234191 36}$$
$$\frac{1}{274877906944}, \frac{1}{549755813888}, 0, \frac{1}{302231454903657293676544}$$
$$\frac{1}{549755813888}, \frac{1}{1099511627776}, 0, \frac{1}{120892581961462917470617 6}$$
$$\frac{1}{1099511627776}, \frac{1}{2199023255552}, 0, \frac{1}{483570327845851669882470 4}$$
$$\frac{1}{2199023255552}, \frac{1}{4398046511104}, 0, \frac{1}{193428131138340667952988 16}$$
$$\frac{1}{4398046511104}, \frac{1}{8796093022208}, 0, \frac{1}{773712524553362671811952 64}$$
$$\frac{1}{8796093022208}, \frac{1}{17592186044416}, 0, \frac{1}{309485009821345068724781056}$$
$$\frac{1}{17592186044416}, \frac{1}{35184372088832}, 0, \frac{1}{123794003928538027489912 4224}$$
$$\frac{1}{35184372088832}, \frac{1}{70368744177664}, 0, \frac{1}{495176015714152109959649 6896}$$
$$\frac{1}{70368744177664}, \frac{1}{140737488355328}, 0, \frac{1}{198070406285660843983859 87584}$$
$$\frac{1}{140737488355328}, \frac{1}{281474976710656}, 0, \frac{1}{792281625142643375935439 50336}$$
$$\frac{1}{281474976710656}, \frac{1}{562949953421312}, 0, \frac{1}{316912650057057350374175801344}$$
$$\frac{1}{562949953421312}, \frac{1}{1125899906842624}, 0, \frac{1}{126765060022822940149670320 5376}$$
$$\frac{1}{1125899906842624}, \frac{1}{2251799813685248}, 0, \frac{1}{507060240091291760598681282 1504}$$
$$\frac{1}{2251799813685248}, \frac{1}{4503599627370496}, 0, \frac{1}{202824096036516704239472512 86016}$$
$$\frac{1}{4503599627370496}, \frac{1}{9007199254740992}, 0, \frac{1}{811296384146066816957890051 44064}$$

$$\frac{1}{9007199254740992}, \frac{1}{18014398509481984}, 0, \frac{1}{3245185536584267267831560205762 56}$$
$$\frac{1}{18014398509481984}, \frac{1}{36028797018963968}, 0, \frac{1}{1298074214633706907132624082305024}$$
$$\frac{1}{36028797018963968}, \frac{1}{72057594037927936}, 0, \frac{1}{5192296858534827628530496329220096}$$
$$\frac{1}{72057594037927936}, \frac{1}{144115188075855872}, 0,$$
$$\frac{1}{2076918743413931051412198531688038 4}$$
$$\frac{1}{144115188075855872}, \frac{1}{288230376151711744}, 0,$$
$$\frac{1}{8307674973655724205648794126752153 6}$$
$$\frac{1}{288230376151711744}, \frac{1}{576460752303423488}, 0,$$
$$\frac{1}{3323069989462289682259517650700861 44}$$
$$\frac{1}{576460752303423488}, \frac{1}{1152921504606846976}, 0,$$
$$\frac{1}{1329227995784915872903807060280344 576}$$
$$\frac{1}{1152921504606846976}, \frac{1}{2305843009213693952}, 0,$$
$$\frac{1}{5316911983139663491615228241121378 304}$$
$$\frac{1}{2305843009213693952}, \frac{1}{4611686018427387904}, 0,$$
$$\frac{1}{2126764793255865396646091296448551 3216}$$
$$\frac{1}{4611686018427387904}, \frac{1}{9223372036854775808}, 0,$$
$$\frac{1}{8507059173023461586584365185794205 2864}$$
$$\frac{1}{9223372036854775808}, \frac{1}{18446744073709551616}, 0,$$
$$\frac{1}{3402823669209384634633746074317682 11456}$$
$$\frac{1}{18446744073709551616}, \frac{1}{36893488147419103232}, 0,$$
$$\frac{1}{1361129467683753853853498429727072 845824}$$
$$\frac{1}{36893488147419103232}, \frac{1}{73786976294838206464}, 0,$$

$$\frac{1}{5444517870735015415413993718908291383296}$$
$$\frac{1}{73786976294838206464}, \frac{1}{147573952589676412928}, 0,$$
$$\frac{1}{217780714829400616616559748756331655331 84}$$
$$\frac{1}{147573952589676412928}, \frac{1}{295147905179352825856}, 0,$$
$$\frac{1}{871122859317602466466238995025326621327 36}$$
$$\frac{1}{295147905179352825856}, \frac{1}{590295810358705651712}, 0,$$
$$\frac{1}{348449143727040986586495598010130648530944}$$
$$\frac{1}{590295810358705651712}, \frac{1}{1180591620717411303424}, 0,$$
$$\frac{1}{139379657490816394634598239204052259412 3776}$$
$$\frac{1}{1180591620717411303424}, \frac{1}{2361183241434822606848}, 0,$$
$$\frac{1}{557518629963265578538392956816209037649 5104}$$
$$\frac{1}{2361183241434822606848}, \frac{1}{4722366482869645213696}, 0,$$
$$\frac{1}{223007451985306231415357182726483615059 80416}$$
$$\frac{1}{4722366482869645213696}, \frac{1}{9444732965739290427392}, 0,$$
$$\frac{1}{892029807941224925661428730905934460239 21664}$$
$$\frac{1}{9444732965739290427392}, \frac{1}{1888946593147858085478 4}, 0,$$
$$\frac{1}{356811923176489970264571492362373784095 686656}$$
$$\frac{1}{1888946593147858085478 4}, \frac{1}{3777893186295716170956 8}, 0,$$
$$\frac{1}{142724769270595988105828596944949513638 2746624}$$
$$\frac{1}{3777893186295716170956 8}, \frac{1}{7555786372591432341913 6}, 0,$$
$$\frac{1}{570899077082383952423314387779798054553 0986496}$$
$$\frac{1}{7555786372591432341913 6}, \frac{1}{1511157274518286468382 72}, 0,$$

$$\left[\begin{array}{ccc} \frac{1}{228359630832953580969325755111919221821239459 84} & \frac{1}{1511157274518286468382 72} & \frac{1}{3022314549036572936765 44} \\ 0, & \frac{1}{913438523331814323877303020447676887284957839 36} & \frac{1}{3022314549036572936765 44} \\ \frac{1}{6044629098073145873530 88} & 0, & \frac{1}{3653754093327257295509212081790707549139831357 44} \\ \frac{1}{6044629098073145873530 88} & \frac{1}{1208925819614629174706 176} & 0, \\ \frac{1}{1461501637330902918203684832716283019655932542976} & \frac{1}{1208925819614629174706 176} & \frac{1}{2417851639229258349412 352} \\ 0, & \frac{1}{5846006549323611672814739330865132078623730171904} & \frac{1}{2417851639229258349412 352} \\ \frac{1}{4835703278458516698824 704} & 0, & \frac{1}{2338402619729444669125895732346052831449492068761 6} \\ \frac{1}{4835703278458516698824 704} & \frac{1}{9671406556917033397649 408} & 0, \\ \frac{1}{9353610478917778676503582929384211325797968275046 4} & & \end{array}\right] \tag{33}$$