MAT331 Exercises, Spring 08 *set number 1*

NOTE: *Each exercise is worth 10 points and can be turned in at any time before its "expiration date". At the end of the semester, I will expect you to have turned in at least 2/5 of the exercises assigned. If you do more, I will pick your best grades. If you do less, the missing grades will be counted as zeros. Altogether, these will count the same as one project.*

Many of these problems will require you to use the help system and/or read the text to figure out what commands you need to use and how to use them.

1. (*expires 2/21*) Use `Maple` to write $x^5 - 2x^4 - 10x^3 + 20x^2 - 16x + 32$ as a product of *exact* linear factors. By exact, I mean you should leave any non-rational factors expressed as radicals; do not approximate terms like $\sqrt{3}$ as 1.73205, etc.

2. (*expires 2/21*) Draw a graph showing both $\cos(x)$ and its fifth Taylor polynomial (that is, $1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4$) for x between -4 and 4. What degree of Taylor polynomial seems to be needed to get good agreement in this range? *Hint: use a variation of the command* `convert(taylor(cos(x),x,5),polynom)` *to make this work. Think of a suitable way to demonstrate that the approximation you have taken is "good"– what is a good definition of "good" here?*

3. (*expires 2/28*) Consider the planar curve γ defined by $x^2y^3 + y^2 + y - 2e^x = 0$. Using **only** `Maple`, find the slope of the tangent line to the curve at $(0, 1)$. Then plot the curve and the tangent line on the same graph.
Hint: you might want to use `implicitplot` *from the library* `plots`*. You might find* `implicitdiff` *helpful, too.*

4. (*expires 2/28*) Plot the function $f(x) = 2\sin x - x^3 - 1/5$, for $x \in [-4, 4]$. Find all the zeros of the function with an accuracy of 20 decimal digits. *Hint: See* `Digits`, `fsolve`.

5. (*expires 2/28*) Define a `Maple` function g that, given a positive integer k yields the sum of the first k primes. What is k such that $g(k) \leq 100,000$ but $g(k+1) > 100,000$? You might find `sum` and `ithprime` helpful.

6. (*expires 2/28*) Use the Taylor expansion of $\arctan x$ near the point $x = 1/\sqrt{3}$ to compute the value of π to 30 places. How many terms are needed to compute the value to 50 places?

7. (*expires 3/4*) Fit the points $(-1.9, -4.7), (-0.8, 1.2), (0.1, 2.8), (1.4, -1.2), (1.8, -3.5)$ by means of a quadratic function $f(x) = ax^2 + bx + c$, using the least square method. First, do this step by step, as we did in class; then, use the built-in `Maple` command, described in the notes. Check that the two solutions agree.

8. (*expires 3/4*) Fit the set of points

$$(1.02, -4.30), (1.00, -2.12), (0.99, 0.52), (1.03, 2.51), (1.00, 3.34), (1.02, 5.30)$$

with a line, using the least square method we used in class. You will see that this is not a good fit. Think of a better way to find a line which *is* a good fit and use `Maple` to do it. Explain in your solution why you think your better way is indeed better.

9. (*expires 3/4*) In this problem we will estimate the charge of the electron.

If an electron of energy E is thrown into a magnetic field B which is perpendicular to its velocity, the electron will be deflected into a circular trajectory of radius r. The relation between these three quantities is:

$$B\,r\,e = \frac{E^2}{m^2c^4}\sqrt{E^2 - m^2c^4}, \tag{1}$$

where e and m are, respectively, the charge and the mass of the electron, and c is the speed of light. The rest mass of the electron is defined as $E_0 = mc^2$, and is about equal to $8.817\;10^{-14}$ Joules. In our experimental set-up the energy of the emitted electrons is set to be $E = 2.511E_0$.

Use `read` to have `Maple` load and execute the commands in the file `electron.txt`, which you can download from the class web page in the problems[1] area, or using the command
`read("/home/scott/www/mat331.spr08/problems/electron.txt")`.

This defines a list called `electron`. Each element of the list is a pair of the form $[B_i, r_i]$, and these quantities are expressed in Teslas and meters. Use least square fitting to determine the best value for e.

[*Hint: Notice that the right hand side of (1) is just a constant—calculate it once and for all and give it a name. Then (1) becomes an equation which is linear in the unknown e. To verify your solution:* $e \approx 1.602\;10^{-19}$ Coulomb].
Physical constants courtesy of N.I.S.T.

[1] http://www.math.sunysb.edu/ scott/mat331.spr08/problems

10. (*expires 3/11*) The set of eight points

$$\begin{array}{cccc} (2.073, 5.794) & (-1.931, 1.316) & (.3959, 3.441) & (3.353, 8.950) \\ (4.267, 12.65) & (3.167, 8.669) & (2.876, 6.992) & (-1.245, 1.616) \end{array}$$

approximate an exponential function of the form $f(x) = ae^{bx}$. Use least squares to find good values for a and b. Plot the data points and your curve on the same axes.

If you don't want to retype the points, you can load them from the file `expdata.txt` from the problems area on the class web page.

NOTE: Neither of the next two problems intrinsically involve `Maple`, except as a word processor to write your solution (although you can use it to help with calculations if you want). If you prefer, you are welcome to turn in a printed or handwritten version instead, though a worksheet is slightly preferred.

11. (*expires 3/11*) Following Section 6 (Fitting a Circle) of the notes, prove that if we describe the circle of center (a, b) and radius r using the parameters (a, b, k), with $k = a^2 + b^2 - r^2$, rather than the more natural parameters (a, b, r), then the error function $H(a, b, k) = E(a, b, \sqrt{a^2 + b^2 - k})$ is quadratic in a, b and k. What does this imply about the number of critical points?

12. (*expires 3/11*) With reference to the previous problem, show that for $r > 0$, the transformation $(a, b, r) \mapsto (a, b, k)$ is a valid change of variables, that is, it is a diffeomorphism (a one-to-one function with continuous nonzero derivatives). This should help you prove that $E(a, b, r)$ has only one "physical" critical point, which is a minimum, and is mapped, through the transformation, into the unique critical point of $H(a, b, k)$.

13. (*expires 3/11*) The set of twelve points

$$\begin{array}{cccc} (-2.256, 0.879) & (-1.764, 5.800) & (-0.684, -0.854) & (-0.776, 6.750) \\ (3.718, 7.394) & (0.081, -1.315) & (-2.357, 4.534) & (6.485, 4.021) \\ (6.518, 3.999) & (2.818, 7.689) & (1.788, -1.668) & (-2.720, 2.719) \end{array}$$

approximate a circle with an unknown radius and center at $(2, 3)$. What is the "best" value for r corresponding to this data? Explain your answer. Plot the resulting circle and the data points on the same graph.

(Note that if you fit the data using the method described in section 6 of the notes, you'll get a somewhat different radius. Using an unknown center gives one at approximately $(1.970, 3.002)$ due to the noise in the data.)

If you don't want to retype the points, you can load them from the file `circdata.txt` from the problems area on the class web page.

14. (*expires 4/1*) Find all the solutions to the differential equation

$$\frac{dx}{dt}(t) = -2x(t), \quad t \in \mathbb{R}.$$

Among them, single out the one for which $x(0) = 3$. [*Hint: read the help page for* `dsolve`*, or just do it in your head. It is that easy.*]

15. (*expires 4/1*) Have `Maple` find analytic solutions to the following system of differential equations,

$$\begin{cases} y''(t) - z(t) &= e^t, \\ z'(t) - y(t) &= 0, \end{cases}$$

with initial conditions: $y(0) = 1,\ y'(0) = 0,\ z(0) = k$. Let us denote the solutions by $y_k(t),\ z_k(t)$ (since they depend on the parameter k).

For k taking all integer values from -10 to 10, and $t \in [-4, 2]$, plot the functions y_k in blue, and the functions z_k in red, all on the same graph. (Yes, you will then have 42 functions plotted on the same graph.) [*This is certainly a case when you don't want to retype the functions that* `Maple` *finds. You will almost certainly need to read the help page for* `dsolve`*. I also found* `subs`*,* `unapply`*, and* `seq` *useful.*]

16. (*expires 4/1*) For the functions $y_k(t)$ and $z_k(t)$ found in the previous problem, plot the parametric curves $\varphi_k(t) = [y_k(t), z_k(t)]$ for integer values of k between -5 and 5 and $-6 < t < 4$ on the same graph. Use the `view` option of plot to only show what lies in the region $-10 < y < 10, -10 < z < 10$, and use a sequence of colors so that each solution is a different color. [*HINT: you might find something like* `seq(COLOR(HUE,i/11),i=0..10)` *useful for the latter.*]

17. (*expires 4/1*) Find all the fixed points of the system

$$\begin{cases} \dot{x} &= x^2 + y, \\ \dot{y} &= x(y^2 - 1), \end{cases}$$

where a "fixed point" is a solution for which **both** $x(t)$ **and** $y(t)$ are constant. For each of these solutions you find, describe the behavior of the solutions that have initial conditions nearby. You can use `Maple` to figure out what happens for nearby points, or you can use more mathematical methods.

NOTE: The fact that there are various notations for differential equations is purely intentional.

18. (*expires 4/29*) Consider the differential equation $\dot{\mathbf{z}}(t) = \mathbf{F}(\mathbf{z}(t))$, where the vector $\mathbf{z}(t) = (x(t), y(t))$ and the field $\mathbf{F}(x, y) = (-y, x - y)$. Plot a few solutions. What happens to them when $t \to +\infty$? Give a "`Maple`-proof" that this is a general fact for *every* solution. [*A "`Maple`-proof" is an argument that is rigorous once we accept `Maple` results as incontrovertibly true.*]

19. (*expires 4/29*) For the equation $\dot{\mathbf{z}} = \mathbf{F}(\mathbf{z})$, $\mathbf{z} = (x, y)$, with the vector field

$$\mathbf{F}(x, y) = \left\langle -x(x^4 + y^4) - y\,,\, x - y(x^4 + y^4) \right\rangle,$$

prove that the origin is an attractor in the future, i.e., every solution verifies

$$\lim_{t \to +\infty} \mathbf{z}(t) = 0.$$

[*While you can ask for assistance in doing this, you must to show clearly that you have understood it.*]

20. (*expires 4/29*) We will study the Lotke-Volterra predator-prey equations: In a very simple ecosystem, at the time t (which is expressed, say, in years), there is a population of $f(t)$ foxes and $r(t)$ rabbits. The evolution of these quantities obeys the system

$$\begin{cases} \dot{f}(t) &= G_f\, f(t) + E\, f(t)\, r(t), \\ \dot{r}(t) &= G_r\, r(t) - E\, f(t)\, r(t); \end{cases}$$

where G_f and G_r are the growth rates for the foxes and the rabbits, respectively, in the absence of each other. E is the probability of a fatal encounter between a fox and a rabbit (normalized per number of foxes and rabbits).

First, write some words to explain why these equations make sense. Then, fix $G_f = 0.4$, $G_r = 2.4$ (it's notorius that rabbits have the tendency to reproduce quickly) and $E = 0.01$. For a few initial conditions of your choice, plot the trajectories in the (f, r)-plane (say, with $0 \leq f \leq 1000$ and $0 \leq r \leq 1000$). For the same initial conditions, plot the actual solutions too (i.e, $f(t)$ against t, and $r(t)$ against t). Write some comments interpreting how the behaviour of the solutions relates to what happens to the two species.

Finally, repeat the same procedure with $G_f = -1.1$. Things change substantially. Again, what is the "physical" interpretation of this?

21. (*expires 5/1*) Write a procedure in `Maple` that counts the frequency of letters in a string of text. For example, here is what it looks like when I use mine:

```
freqs("time flies like an arrow, fruit flies like a bananna.");
```

```
[[" ",9], ["i",6], ["a",6], ["e",5], ["n",4], ["l",4],
 ["r",3], ["f",3], ["t",2], ["s",2], ["k",2], ["w",1],
 ["u",1], ["o",1], ["m",1], ["b",1], [",",1], [".",1]]
```

In the above phrase, there are 9 spaces, 6 each of the letters "i" and "a", "e" appears 5 times, and so on. [Hint: I found it useful to group identical letters in the text using `Implode(sort(Explode(text)))`, but you might not.]

22. (*expires 5/1*) The text below was encrypted with a substitution cipher. Only the letters (both upper-case and lower-case) were substituted, leaving punctuation and spaces alone. Figure out what the original message was.

> "wA'r aBeD WUeK AP XNaB NM U rALKNP USUeAJBMA NM zUM nPrB ZNAW U JUM ZWP'r XBUeMNMO AP SXUD AWB aNPXNM." yWUA'r ZWUA rWB APXK AWB SPXNCB ZWBM rWB WUMKBK AWBJ AWB BJSAD eBaPXaBe.
>
> lNCWUeK heULANOUM, "yWB zCUeXUAAN yNXA"

If you wish, you can find the encrypted text in the file `subscrypt.txt` from the problems area on the class web page.

23. (*expires 5/1*) The cryptography chapter in the notes is called "fsqFsHn sGGousG", which is actually the result of applying a Caesar cipher to its original title. A 53-character alphabet consisting of all the upper-case letters, a space, and all the lower-case letters was used; consequently the space in the middle might or might not correspond to a space in the title. Determine what the original title was.

24. (*expires 5/13*) The string below was encrypted using an affine cipher on the 27 letter alphabet " abcdefghijklmnopqrstuvwxyz" (there is a space in the 0^{th} position.) Decrypt it.

```
fmw segjaweoouanerj a ceyqrype aswaheoaqbrqabeafrua eeaojerf afmjeayperjpu
```

Hint: this phrase follows the the typical pattern in English where there are (almost) as many spaces as words (and so spaces are very common), and the letter "e" is also very common. You can use the technique described in chapter 4 of the notes, section 7.3.

25. (*expires 5/13*) Recall that a Vignère cipher can be interpreted as a Caesar-like cipher on n-vectors, where n is the length of the key phrase. Can every affine encipherment on digraphs (two-character codes) be interpreted as an affine matrix encipherment on 2-vectors? That is, suppose I encode a message by affine enciphering on digraphs. Can I always get the same crypttext from the same plaintext using an affine matrix enciphering (using a 2×2 matrix) on 2-vectors? If your answer is yes, prove it. If no, give a counter-example that cannot be so interpreted.

26. (*expires 5/13*) Modify the `AffineMatEncode` routine from the notes so that you can use a text string as a key instead of a matrix and a vector. For example, if the phrase is k characters long, the key should be an $n \times n$ matrix and an n-vector, where $n^2 + n \approx k$. The elements of the key matrix and vector should be the numerical equivalents of the characters in the key phrase. Do something sensible with any extra letters (that is, if $k \neq n^2 + n$). Be sure to check that the resulting matrix is nonsingular.

27. (*expires 5/21*) Reimplement the `AffineMatEncode` and `AffineMatDecode` programs from the notes (section 4.8) to use the routines from the `LinearAlgebra` package instead of `linalg`.

28. (*expires 5/21*) Twenty-one pirates are dividing their horde of gold dubloons. Since they are a democratic outfit, they first try to divide the coins evenly, but they find there are 19 coins left over. The "discussion" about how to divide the remaining coins results in only 16 pirates still needing to divide the horde (the remaining five went to a place where you can't bring money or anything else with you). The redivision among 16 pirates leaves 1 coin left over, and three of the pirates make a grab for it. These three find themselves to be missing their hands after this attempt, and the remaining thirteen pirates decide to divide the share among themselves, leaving the handless ones with nothing. Fortunately, the horde divides evenly among the thirteen. What is the minimum number of coins in the horde?

29. (*expires 5/21*) Use RSA with the modulus $n = 119$ and the exponent $e = 7$, with the 95-character alphabet consisting of the printable ASCII characters to encrypt the word "Yes". Recall that the alphabet is given by

```
Alphabet:=cat(op(select(IsPrintable, [seq( convert([i],bytes), i=1..255)])))
```

so that Y=58, e=70, s=84. Give your encryption as a list of three numbers.

30. (*expires 5/21*) With the same setup as the previous problem (that is $n = 119$, $e = 7$), the message after encrypting with RSA is the list of numbers

$$[42, 59, 4, 59, 27, 59].$$

Decrypt the message. (This is doable because 119 is easily factored).