Homework Assignment 9

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Exercise 1. In each of the following cases, let T be the linear operator on $\mathbb{R}^2$ which is represented by the matrix A in the standard ordered basis for $\mathbb{R}^2$, and let U be the linear operator on $\mathbb{C}^2$ represented by A in the standard ordered basis. Find the characteristic polynomial for T and that for U, find the characteristic values of each operator, and for each such characteristic value c find a basis for the corresponding space of characteristic vectors.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad A = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Solution: In all cases, denote by $\mathcal{B}_c$ the basis for the subspace corresponding to the characteristic value c

First matrix The characteristic polynomial is $x(x-1)$. The roots are 0 and 1. $\mathcal{B}_0 = \{(0,1)\}$ and $\mathcal{B}_1 = \{(1,0)\}$. In this case the real and complex cases are the same.

Second matrix The characteristic polynomial is $5 - 3x + x^2$ which has no real roots. The complex eigenvalues are $\frac{3}{2} + i\frac{1}{2}\sqrt{11}$, $\frac{3}{2} - i\frac{1}{2}\sqrt{11}$, $\mathcal{B}_{\frac{3}{2}+i\frac{1}{2}\sqrt{11}} = \{(1, -\frac{1}{6} + i\frac{1}{6}\sqrt{11})\}$ and $\mathcal{B}_{\frac{3}{2}-i\frac{1}{2}\sqrt{11}} = \{(1, -\frac{1}{6} - i\frac{1}{6}\sqrt{11})\}$

Third matrix The characteristic polynomial is $x^2 - 2x$. The roots are 0 and 2. $\mathcal{B}_0 = \{(-1,1)\}$ and $\mathcal{B}_2 = \{(1,1)\}$. In this case the real and complex cases are the same.

Exercise 4. Let T be the linear operator on $\mathbb{R}^3$ which is represented in the standard ordered basis by the matrix

$$\begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix}.$$

Prove that T is diagonalizable by exhibiting a basis for $\mathbb{R}^3$, each vector of which is a characteristic vector of T.

Solution: The characteristic polynomial of T is $(x+1)^2(x-3)$. The characteristic values are -1 and 3. $\mathcal{B}_{-1} = \{(0,1,-1),(1,0,2)\}$ and $\mathcal{B}_3 = \{(1,1,2)\}$. $B_{-1} \cup B_3$ is a basis for $\mathbb{R}^3$.

Exercise 5. Let

$$A = \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix}$$

Is A similar over the field $\mathbb{R}$ to a diagonal matrix? Is A similar over the field $\mathbb{C}$ to a diagonal matrix?

Solution: The characteristic polynomial of A is $-2+x-2x^2+x^3$ which has roots

2, i and $-i$. Therefore A is not similar over $\mathbb{R}$ to a diagonal matrix, by theorem 2. Also by theorem 2 we get that A *is* similar to a diagonal matrix over $\mathbb{C}$.

Exercise 8. Let A and B be $n \times n$ matrices over the field F. Prove that if $I - AB$ is invertible, then $I - BA$ is invertible and

$$(I - BA)^{-1} = I + B(I - AB)^{-1}A$$

Solution: Using the expression we are given, we get:

$$(I - BA)(I + B(I - AB)^{-1}A) = I - BA + B(I - AB)^{-1}A - BAB(I - AB)^{-1}A$$

After the I we can factor B on the left and A on the right, and we get:

$$(I - BA)(I + B(I - AB)^{-1}A) = I - B[-I + (I - AB)^{-1} - AB(I - AB)^{-1}]A = \\ I - B[-I + (I - AB)(I - AB)^{-1}]A = I + 0 = I$$

Exercise 9. Use the result of Exercise 8 to prove that, if A and B are $n \times n$ matrices over the field F, then AB and BA have the same characteristic values in F.

Solution: We have to show that if x is a characteristic value for AB then x is a characteristic value for BA (and conversely). This is equivalent to the statement, if x is *not* a characteristic value for BA then it is *not* a characteristic value for AB. We will prove this last statement.

Suppose that x is not a characteristic value for BA, this means that $\det(xI - BA) \neq 0$. There are two cases:

Case 1: $x = 0$. In this case $\det(-BA) \neq 0$. But $\det(-BA) = (-1)^n \det(B)\det(A) = (-1)^n \det(A)\det(B) = \det(-AB) = \det(xI - AB)$. Therefore $\det(xI - AB) \neq 0$.

Case 2: $x \neq 0$. In this case $xI - BA = x(I - \frac{1}{x}BA)$ and $\det(x(I - \frac{1}{x}BA)) = x^n \det(I - \frac{1}{x}BA) \neq 0$. Therefore $I - \frac{1}{x}BA$ is invertible, but this implies (by the previous exercise) that $I - A\frac{1}{x}B = I - \frac{1}{x}AB$ is invertible, therefore $\det(I - \frac{1}{x}AB) \neq 0$, therefore $x^n \det(I - \frac{1}{x}AB) = \det(xI - AB) \neq 0$.

Exercise 13. Let V be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$ that are continuous, i.e. the space of all continuous real-valued functions on the real line. Let T be the linear operator on V defined by

$$T(f) = \int_0^x f(t)dt.$$

Prove that T has no characteristic values.

Solution: Suppose that T has a characteristic value, i.e.

$$\int_0^x f(t)dt = cf(x)$$

for some f not identically zero. Notice that if $c = 0$ then f is identically zero by the mean value theorem *why?*. Therefore $c \neq 0$. Define $x_0 = \sup x \in \mathbb{R} \mid f(x) = 0$. If $x_0 < \infty$, by continuity there exists $\epsilon > 0$ such that $f(x) > 0$ (or $f(x) < 0$) if $x_0 < x < x + \epsilon$. In that neighborhood we have the differential equation

$$f'(x) = cf(x)$$

with initial condition $f(x_0) = 0$. This equation has the solution $f(x) = Ke^{\frac{1}{c}x}$. But the initial condition implies that $K = 0$. Therefore $f(x) \equiv 0$ in the neighborhood $x_0 \leq x \leq x_0 + \epsilon$. Therefore $x_0 = \infty$ i.e. $f \equiv 0$ on the positive real axis. Analogously $f \equiv 0$ on the negative real axis. This contradicts the supposition that f is a characteristic vector.