Homework assignment 8

Section 3.7 pp. 115

Exercise 2.

Let V be the vector space of all polynomial functions over the field of real numbers. Let a and b be fixed real numbers and let f be the linear functional on V defined by

$$f(x) = \int_a^b p(x)dx$$

If D is the differentiation operator on V, what is $D^t f$?

Solution: By the fundamental theorem of calculus, we get:

$$D^t f(q) = f(D(q)) = \int_a^b q'(x)dx = q(b) - q(a)$$

Exercise 3. Let V be the space of all $n \times n$ matrices over a field F and let B be a fixed $n \times n$ matrix. If T is the linear operator on V defined by $T(A) = AB - BA$ and if f is the trace function, what is $T^t f$?

Solution: Using the result from the previous hw. $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ we get:

$$T^t f(A) = f(T(A)) = \mathrm{tr}(T(A)) = \mathrm{tr}(AB - BA) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = 0$$

Exercise 6. Let n be a positive integer and let V be the space of all polynomial functions over the field of real numbers which have degree at most n, i.e., functions of the form

$$f(x) = c_0 + \ldots + c_n x^n$$

Let D be the differentiation operator on V. Find a basis for the null space of the transpose D^t.

Solution: Since $(D^t)^t = D$ (not equal, but canonically identified, $\dim V < \infty$), using theorem 22 we get that the range of $D = (D^t)^t$ is the annihilator of the null space of D^t. But we know from the last Hw that the range of D consists of all polynomials of degree *strictly* less than n. Therefore, the null space of D^t consists of all functionals which vanish on polynomials which only contain terms of degree n. A basis for this space is $\{f\}$ where $f(c_0 + \cdots + c_n x^n) = c_n$. The explicit calculation done in the recitation yields the same result.

Exercise 7. Let V be a finite dimensional vector space over the field F. Show that $T \mapsto T^t$ is an isomorphism of $L(V, V)$ onto $L(V^*, V^*)$.

Solution: Let $\phi : L(V, V) \to L(V^*, V^*)$ be given by $\phi(T) = T^t$. Using the definition of the transpose we get that ϕ is linear. At this point we can either check that ϕ is one to one and onto or find an inverse for ϕ. Let us do the latter. Using the canonical identification of V with $(V^*)^*$ (here is where we use the fact that V is finite dimensional) we define $\psi : L(V^*, V^*) \to L((V^*)^*, (V^*)^*) = L(V, V)$ by the formula $\psi(S) = S^t$. Using the definitions and the canonical identification we get that $\psi \circ \phi = Id$ and $\phi \circ \psi = Id$

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Exercise 5. Let A be a 2×2 matrix over a field F., and suppose that $A^2 = 0$. Show for each scalar c that $\det(cI - A) = c^2$

Solution: Multiplying $cI - A$ times A we get $A(cI - A) = cA - A^2 = cA$, therefore

$$\det(A)\det(cI - A) = c^2 \det(A)$$

so if $\det(A) \neq 0$ we get the result. Suppose now that $\det(A) = 0$ and $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Then $\det(cI - A) = c^2 - c(x + w) + \det(A) = c^2 - c(x + w)$. Notice that since $\det(A) = 0$ $0 = \mathrm{tr}(A^2) = (x + w)^2$ therefore $x + w = 0$ and we get the identity.

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Exercise 3. An $n \times n$ matrix A over a field F is **skew-symmetric** if $A^t = -A$. If A is a skew-symmetric $n \times n$ matrix with complex entries and n is odd, prove that $\det(A) = 0$

Solution: We know that $\det(A) = \det(A^t)$ and that $\det(cA) = c^n \det(A)$. Thus, since n is odd $\det(A) = \det(A^t) = \det(-A) = \det((-1)A) = (-1)^n \det(A) = -\det(A)$ therefore $2\det(A) = 0$, but over $\mathbb{C}$ this implies that $\det(A) = 0$

Exercise 4. An $n\times n$ matrix over a field F is called **orthogonal** if $AA^t = I$. If A is orthogonal show that $\det(A) = \pm 1$. Give an example of an orthogonal matrix for which $\det(A) = -1$

Solution: Since $\det(A) = \det(A^t)$, we get $\det(I) = \det(AA^t) = \det(A)\det(A^t) = \det(A)^2 = 1$. Therefore $\det(A) = \pm 1$. An example with determinant equal to -1 is:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Exercise 5. An $n \times n$ matrix A over the field of complex numbers is said to be **unitary** if $AA^* = I$ (A^* denotes the conjugate transpose of A). If A is unitary, show that $|\det(A)| = 1$.

Solution: Notice that if $p(x)$ is a complex polynomial $p(\bar{x}) = \overline{p(x)}$ (taking the conjugate commutes with taking sums and products). Therefore $\det(\overline{A}) = \overline{\det(A)}$ (the determinant is a polynomial in the entries of the matrix). Thus, we get $\det(A^*) = \det(\overline{A}^t) = \det(\overline{A}) = \overline{\det(A)}$. This implies that $1 = \det(I) = \det(AA^*) = \det(A)\det(A^*) = \det(A)\overline{\det(A)} = |\det(A)|$.