Homework Assignment 6

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Exercise 1. Let T be the linear operator on $\mathbb{C}^2$ defined by $T(x_1, x_2) = (x_1, 0)$. Let $\mathcal{B}$ be the standard ordered basis for $\mathbb{C}^2$ and let $\mathcal{B}' = \{\alpha_1, \alpha_2\}$ be the ordered basis defined by $\alpha_1 = (1, i),\ \alpha_2 = (-i, 2)$.

a What is the matrix of T relative to the pair $\mathcal{B}, \mathcal{B}'$?

b What is the matrix of T relative to the pair $\mathcal{B}', \mathcal{B}$?

c What is the matrix of T in the ordered basis $\mathcal{B}'$?

d What is the matrix of T in the ordered basis $\{\alpha_2, \alpha_1\}$?

Solution:

a

$$M = \begin{pmatrix} 2 & 0 \\ -i & 0 \end{pmatrix}$$

b

$$M = \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}$$

c

$$M = \begin{pmatrix} 2 & -2i \\ -i & -1 \end{pmatrix}$$

d

$$\begin{pmatrix} -1 & -i \\ -2i & 2 \end{pmatrix}$$

Exercise 4. Let V be a two-dimensional vector space over the field F, and let $\mathcal{B}$ be an ordered basis for V. If T is a linear operator on V and

$$[T]_{\mathcal{B}} = M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

prove that $T^2 - (a+d)T + (ad - bc)I = 0$.

Solution: Notice that the zero operator and the identity operator have the same matrix with respect to any basis, namely, the zero matrix and the identity matrix. Therefore we only need to check the equality with respect to the basis $\mathcal{B}$.

$$M^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ca + dc & bc + d^2 \end{pmatrix}, \quad (a+d)M = \begin{pmatrix} a^2 + da & ba + bd \\ ac + cd & ad + d^2 \end{pmatrix}, \quad (ad-bc)I = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix}$$

Therefore $M^2 - (a+d)M + (ad-bc)I = 0$.
Exercise 8 Let θ be a real number. Prove that the following two matrices are similar over the field of complex numbers:

$$\begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t)t & \cos(t) \end{pmatrix}, \quad \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}.$$

(*Hint:* Let T be the linear operator on $\mathbb{C}^2$ that is represented by the first matrix in the standard ordered basis. Find vectors α_1 and α_2 such that $T\alpha_1 = e^{it}\alpha_1$, $T\alpha_2 = e^{-it}\alpha_2$, and $\{\alpha_1, \alpha_2\}$ is a basis.)

Solution: If there exists a basis α_1, α_2 such that $T\alpha_1 = e^{it}\alpha_1$, $T\alpha_2 = e^{-it}\alpha_2$, then T is represented in this basis by the second matrix. It means that the second matrix is similar to the first. So it remains to find α_1, α_2. Let $\alpha_1 = (x_1, x_2)$. Then $T\alpha_1 = (\cos(t)x_1 - \sin(t)x_2, \sin(t)x_1 + \cos(t)x_2)$. Hence we arrive to the system of equations

$$\cos(t)x_1 - \sin(t)x_2 = e^{it}x_1$$
$$\sin(t)x_1 + \cos(t)x_2 = e^{it}x_2,$$

or equivalently

$$(\cos(t) - e^{it})x_1 - \sin(t)x_2 = 0$$
$$\sin(t)x_1 + (\cos(t) - e^{it})x_2 = 0.$$

Since $e^{it} = \cos(t) + i\sin(t)$, we get

$$-i\sin(t)x_1 - \sin(t)x_2 = 0$$
$$\sin(t)x_1 - i\sin(t)x_2 = 0.$$

So we can divide by $\sin(t)$ and find that $x_1 = ix_2$, i.e. we can take $\alpha_1 = (i, 1)$. Similar calculations allow to find α_2.

Exercise 9. Let V be a finite dimensional vector space over the field F and let S and T be linear operators on V. We ask: When do there exist ordered bases $\mathcal{B}$ and $\mathcal{B}'$ for V such that $[S]_{\mathcal{B}} = [T]_{\mathcal{B}'}$? Prove that such bases exist if and only if there is an invertible linear operator U on V such that $T = USU^{-1}$.
Solution: Suppose that $[S]_{\mathcal{B}} = [T]_{\mathcal{B}'}$, where $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ and $\mathcal{B}' = \{\beta_1, \ldots, \beta_n\}$. Let U be the operator which carries $\mathcal{B}$ onto $\mathcal{B}'$. Let v be any vector of V. Then $v = a_1\alpha_1 + \ldots + a_n\alpha_n$ for some unique scalars $a_1, \ldots, a_n$. Let $w = U(v) = a_1\beta_1 + \ldots + a_n\beta_n$ Since $[S]_{\mathcal{B}} = [T]_{\mathcal{B}'}$ we know that $S(v) = c_1\alpha_1 + \ldots + c_n\alpha_n$ and $T(w) = c_1\beta_1 + \ldots + c_n\beta_n$ Therefore

$$U^{-1}TU(v) = UT(w) = U(c_1\beta_1 + \ldots + \gamma_n\beta_n) = c_1\alpha_1 + \ldots + c_n\alpha_n = S(v)$$

Now suppose that there exists an invertible operator U such that $T = USU^{-1}$. Let $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ be any basis for V and let $\mathcal{B}' = \{U(\alpha_1), \ldots, U(\alpha_n)\}$. Then, if $v = a_1\alpha_1, \ldots, a_n\alpha_n$ is any vector in V,

$$T(v) = T(a_1\alpha_1, \ldots, a_n\alpha_n) = c_1\alpha_1 + \ldots + c_n\alpha_n = USU^{-1}(v)$$

but $U^{-1}(v) = a_1\beta_1 + \ldots + a_n\beta_n$. Hence, the matrix representing S in the basis $\mathcal{B}$ has the same entries as the matrix representing T in the basis $\mathcal{B}'$.

Bonus exercise 12. Let V be an n-dimensional space over the field F, and $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ an ordered basis for V.

(a) There is a unique linear operator T on V such that

$$T\alpha_i = \alpha_{i+1}, \quad j = 1, \ldots, n-1, \quad T\alpha_n = 0.$$

What is the matrix A of T in the ordered basis $\mathcal{B}$?

(b) Prove that $T^n = 0$ but $T^{n-1} \neq 0$.

(c) Let S be any linear operator on V such that $S^n = 0$ but $S^{n-1} \neq 0$. Prove that there is an ordered basis $\mathcal{B}'$ for V such that the matrix of S in the ordered basis $\mathcal{B}'$ is the matrix A of part (a).

(d) Prove that if M and N are $n \times n$ matrices over F such that $M^n = N^n = 0$ but $M^{n-1} \neq 0 \neq N^{n-1}$, then M and N are similar.

Solution: (a)

$$A = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \ldots & 1 & 0 & 0 \\ 0 & 0 & \ldots & 0 & 1 & 0 \end{pmatrix}$$

(b) The operator T^{n-1} maps α_1 to α_n. Indeed, $T^{n-1}\alpha_1 = T^{n-2}(T\alpha_1) = T^{n-2}\alpha_2 = \ldots = \alpha_n$. Hence, $T^{n-1} \neq 0$. But the same arguments show that T^n maps any basis vector α_i to 0, hence $T^n = 0$.

(c) Let $\alpha \in V$ be a vector such that $T^{n-1}\alpha \neq 0$. Then vectors $\beta_1 = \alpha$, $\beta_2 = T\alpha$,..., $\beta_n = T^{n-1}\alpha$ form a basis in V. These vectors are linearly independent because if $a_1\beta_1 + \ldots + a_n\beta_n = 0$, then applying T^{n-1} we get rid of all terms except for $a_1(T^{n-1}\alpha) = 0$. Hence, $a_1 = 0$. Applying T^{n-2} we get that $a_2(T^{n-1}\alpha) = 0$, hence $a_2 = 0$ and so on. Then $\mathcal{B}' = \{\beta_1, \ldots, \beta_n\}$ is the desired basis.

(d) Consider an operator whose matrix in some basis is M. Then by (c) there exists a basis such that this operator has matrix A from (a) in this basis. Hence, M is similar to A. The same is true for N. And two matrices that are similar to the same matrix are similar to each other.