Homework Assignment 5

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Exercise 1. Which of the following maps T from $\mathbb{R}^2$ into $\mathbb{R}^2$ are linear transformations?

(a) $T(x_1, x_2) = (1 + x_1, x_2)$; No, because $T(0,0) \neq (0,0)$.
(b) $T(x_1, x_2) = (x_2, x_1)$; Yes, because x_2 and x_1 are linear homogeneous functions of x_1, x_2.
(c) $T(x_1, x_2) = (x_1^2, x_2)$; No, because, say, $2T(1,0) \neq T(2,0)$.
(d) $T(x_1, x_2) = (\sin x_1, x_2)$; No, since $2T(\frac{\pi}{2}, 0) = (2,0) \neq (0,0) = T(\pi, 0)$.
(e) $T(x_1, x_2) = (x_1 - x_2, 0)$. Yes, because $x_1 - x_2$ and 0 are linear homogeneous functions of x_1, x_2.

Exercise 3. Find the range, rank, null space, and nullity for th e differentiation transformation D on the space of polynomials of degree $\leq k$:

$$D(f) = f'.$$

Do the same for the integration transformation T:

$$T(f) = \int_0^x f(t)dt.$$

Solution: The range of D consists of all polynomials of degree strictly less than k, since any polynomial $p(x) = a_n x^n + \ldots + a_0$ is the derivative of the polynomial $\int p(x) = \frac{a_n}{n+1} x^{n+1} + \ldots + a_0 x$. The null space of D consists of all constants. Hence, the rank of D is k, and the nullity is 1.

The range of T consists of all continuous functions f such that f has *continuous* first derivative and $f(0) = 0$. The null space of T is trivial, because if a function is not identically zero then so is its integral. Hence, the rank of T is infinite, and the nullity is 0.

Exercise 7. Let F be a subfield of the complex numbers and let T be the function from F^3 into F^3 defined by

$$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3)$$

(a) Verify that T is a linear transformation.

(b) If (a, b, c) is a vector in F^3, what are the conditions on a, b and c that the vector be in the range of T? What is the rank of T?

(c) What are the conditions on a, b, and c that (a, b, c) be in the null space of T? What is the nullity of T?

Solution:

(a) The coordinate functions of T are given by homogeneous polynomials of degree 1.

(b) Let

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{pmatrix}$$

that is, A is the matrix which represents T with respect to the canonical basis of F^3. If we row reduce A^T we obtain

$$\tilde{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

therefore, a basis for the image of T is $(1,0,1),(0,1,-1)$, i.e. (a,b,c) is in the range of T if and only if there are scalars $s,t \in F$ such that

$$(a,b,c) = s(1,0,1) + t(0,1,-1)$$

i.e. The rank of T is 2

(c) The conditions for (a,b,c) to be in the kernel are

$$a = -\frac{2}{3}c, \quad b = \frac{4}{3}c$$

The nullity of T is 1 by the dimension formula.

Exercise 8. Describe explicitly the linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^3$ that has as its range the subspace spanned by $(1,0,-1)$ and $(1,2,2)$.

Solution: E.g. one can take the transformation T that takes $(1,0,0)$ to $(1,0,-1)$, $(0,1,0)$ to $(1,2,2)$ and $(0,0,1)$ to $(0,0,0)$. Explicitly

$$T(x_1,x_2,x_3) = (x_1 + x_2, 2x_2, -x_1 + 2x_2)$$

Exercise 9. Let V be the vector space of all $n \times n$ matrices over the field F, and let B be a fixed $n \times n$ matrix. If

$$T(A) = AB - BA$$

verify that T is a linear transformation from V to V.

Solution: By the definition of T:

$$T(A_1 + cA_2) = (A_1 + cA_2)B - B(A_1 + cA_2)$$

As in example 4 we conclude

$$(A_1 + cA_2)B - B(A_1 + cA_2) = A_1B + cA_2B - BA_1 + cA_2 = (A_1B - BA_1) + c(A_2B - BA_2)$$

but this last expression is equal to

$$T(A_1) + cT(A_2)$$

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Exercise 2. Let T be the unique linear operator on $\mathbb{C}^3$ for which

$$Te_1 = (1,0,i), \quad Te_2 = (0,1,1), \quad Te_3 = (i,1,0)$$

is T invertible?

Solution. The matrix which represents T with respect to the canonical basis is

$$A_T = \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 1 \\ i & 1 & 0 \end{pmatrix}$$

Therefore T is invertible if and only if the matrix A_T is invertible. But, notice that row 3 of this matrix is the sum of row 2 and i times row 1. This means T cannot be invertible (because A_T is not of full rank).

Exercise 5. Let $\mathbb{C}^{2\times 2}$ be the complex vector space of 2×2 matrices with complex entries. Let

$$B = \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix}$$

and let T be the linear operator on $\mathbb{C}^{2\times 2}$ defined by $T(A) = BA$. What is the rank of T? can you describe T^2?

Solution: Let

$$\mathcal{B} = \left\{ e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$\mathcal{B}$ is a basis for $\mathbb{C}^3$. Since $T(e_1) = -T(e_3)$ and $T(e_2) = -T(e_4)$ we conclude that the rank of T is less than or equal to 2. Since $T(e_1)$ and $T(e_2)$ are linearly independent. The rank is 2. Notice that $B^2 = 5B$ therefore $T^2(A) = B^2A = 5BA = 5T(A)$

Exercise 7. Find two linear operators T and U on $\mathbb{R}^2$ such that $TU = 0$ but $UT \neq 0$.

Solution: Take $T(x_1, x_2) = (x_2, 0)$ and $U(x_1, x_2) = (0, x_2)$

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Exercise 2. Let V be a vector space over the field of complex numbers, and suppose there is an isomorphism T of V onto $\mathbb{C}^3$. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be vectors in V such that

$$T\alpha_1 = (1,0,i), \quad T\alpha_2 = (-2, 1+i, 0),$$
$$T\alpha_3 = (-1,1,1), \quad T\alpha_4 = (\sqrt{2}, i, 3).$$

(a) Is α_1 in the subspace spanned by α_2 and α_3?

(b) Let W_1 be the subspace spanned by α_1 and α_2, and let W_2 be the subspace spanned by α_3 and α_4. What is the intersection of W_1 and W_2?

(c) Find a basis for the subspace of V spanned by the four vectors α_j.

Solution: (a) Note that since T is an isomorphism, it is one-to-one. So α_1 is in the subspace spanned by α_2 and α_3 if and only if $T\alpha_1$ is in the subspace spanned by $T\alpha_2$ and $T\alpha_3$. It is easy to find that $T\alpha_1 = (1, 0, i) = -\frac{1+i}{2}(-2, 1+i, 0) + i(-1, 1, 1) = -\frac{1+i}{2}T\alpha_2 + iT\alpha_3$. Hence, $T\alpha_1$ belongs to the subspace spanned by $T\alpha_2$ and $T\alpha_3$.

(b) The intersection of W_1 and W_2 is the image of the intersection TW_1 and TW_2 under the action of T^{-1}. So first, find $TW_1 \cap TW_2$. Since we already know from the part (a) that $T\alpha_1 + \frac{1+i}{2}T\alpha_2 = -iT\alpha_3$, we get that $T\alpha_3$ does belong to TW_1. On the other hand, it is easy to check that $T\alpha_4$ does not. Indeed, if $a(1, 0, i) + b(-2, 1+i, 0) = (\sqrt{2}, i, 3)$, then $ai = 3$ and $b(1+i) = i$, but then $a - 2b \neq \sqrt{2}$. Hence, the intersection $TW_1 \cap TW_2$ is spanned by $T\alpha_3$, and the intersection $W_1 \cap W_2$ is spanned by α_3.

(c) From parts (a) and (b) we know that α_3 lies in the subspace spanned by α_1, α_2, but α_4 does not. Hence, vectors α_1, α_2, a_4 are linearly independent and span any of the four vectors α_j. So they form a basis for the subspace of V spanned by the four vectors α_j. Note that this subspace coincides with V itself, since they both have dimension 3.

Exercise 4. Show that $F^{m\times n}$ (the space of $m \times n$ matrices) is isomorphic to F^{mn} (the mn-tuple space).

Solution: Denote by E_{ij}, where $1 \leq i \leq m, 1 \leq j \leq n$, the $m \times n$ matrix whose only nonzero entry is $(E_{ij})_{ij} = 1$. We have mn such matrices. I claim that they form a basis in $F^{m\times n}$. Indeed, consider their arbitrary linear combination with coefficients a_{ij}. We get the $m \times n$ matrix with entries a_{ij}. This matrix is zero if and only all coefficients are zero, so E_{ij}s are linearly independent. On the other hand, any $m \times n$ matrix with arbitrary entries b_{ij} is the linear combination of E_{ij} with coefficients b_{ij}, so E_{ij}s span $F^{m\times n}$.

Let e_i, where $1 \leq i \leq mn$, be the standard basis in F^{mn}. Then it is easy to check that the linear operator from $F^{m\times n}$ to F^{mn} that takes E_{ij} to $e_{(i-1)n+j}$ is an isomorphism.

Bonus exercise 7. Let V and W be vector spaces over the field F and let U be an isomorphism of V onto W. Prove that $T \to UTU^{-1}$ is an isomorphism of $L(V, V)$ onto $L(W, W)$ (here $L(V, V)$ is the space of all linear operators from V to V).

Solution: Let T be an operator on V. Then the composition

$$UTU^{-1} : W \overset{U^{-1}}{\to} V \overset{T}{\to} V \overset{U}{\to} W$$

is the operator on W. So the map $\mathcal{T} : T \to UTU^{-1}$ takes an operator on V to the operator on W. Clearly, this map is linear, since $U(T + cT')U^{-1} = UTU^{-1} + cUT'U^{-1}$.

Let us prove that it is invertible. Consider the map $\mathcal{S} : L(W, W) \to L(V, V)$ that takes an operator S on W to the operator

$$U^{-1}SU : V \overset{U}{\to} W \overset{S}{\to} W \overset{U^{-1}}{\to} V$$

on V. Then $\mathcal{TS} = \mathcal{ST} = I$, since $U(U^{-1}SU)U^{-1} = S$ and $U^{-1}(UTU^{-1})U = T$.