# Homework Assignment 2

p 21

**Exercise 2.** Let

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{pmatrix}$$

Verify directly that $A(AB) = A^2B$

**Solution:**

$$A^2 = \begin{pmatrix} 2 & -1 & 1 \\ 5 & -2 & 3 \\ 6 & -3 & 4 \end{pmatrix}, \quad A^2B = \begin{pmatrix} 7 & -3 \\ 20 & -4 \\ 25 & -5 \end{pmatrix}$$

$$AB = \begin{pmatrix} 5 & -1 \\ 8 & 0 \\ 10 & -2 \end{pmatrix}, \quad A(AB) = \begin{pmatrix} 7 & -3 \\ 20 & -4 \\ 25 & -5 \end{pmatrix}$$

**Exercise 3.** Find two different $2 \times 2$ matrices $A$ such that $A^2 = 0$ but $A \neq 0$.

**Solution:** Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a $2 \times 2$ matrix. Then

$$A^2 = \begin{pmatrix} a^2 + bc & (a+d)b \\ (a+d)c & d^2 + bc \end{pmatrix}.$$

Now find $a, b, c, d$ such that $a^2 + bc = (a+d)b = (a+d)c = d^2 + bc = 0$. Note that if $a + d = 0$ and $ad - bc = 0$, then all these equations are satisfied. For instance, put $a = -d = 1, b = -c = 1$ or put $a = d = 0, b = 0, c = 1$. Then the matrices

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

are not $0$ but their squares are $0$.

**Exercise 4.** For the matrix $A$ of Exercise $2$, find elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k \cdots E_2 \cdot E_1 \cdot A = I$$

**Solution:** Let us first row-reduce $A$ into the identity matrix:

$$A \xrightarrow{-2\cdot I+II} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 3 & 0 & 1 \end{pmatrix} \xrightarrow{-3\cdot I+III} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 3 & -2 \end{pmatrix} \xrightarrow{\frac{1}{2}\cdot II}$$

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 3 & -2 \end{pmatrix} \xrightarrow{-3\cdot II+III} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \xrightarrow{-2\cdot III} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\frac{1}{2}\cdot III+II}$$

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{-1\cdot III+I} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{1\cdot II+I} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Applying the transformation on top of each arrow to the identity matrix, we obtain the elementary transformations we want. For example, $E_1$ is obtained by applying $-2 \cdot I + II$ to the identity matrix, therefore:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Analogously we obtain all the other matrices, the last one is

$$E_8 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

**Exercise 7.** Let $A$ and $B$ be $2 \times 2$ matrices such that $AB = I$. Prove that $BA = I$.

**Solution:**

*First proof:* Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

Consider $x_i$ as unknowns, and $a, b, c, d$ as coefficients. Since $AB = I$, the unknowns $x_1, x_3$ satisfy the following 2 linear equations

$$\begin{aligned} ax_1 &+ bx_3 &= 1 \\ cx_1 &+ dx_3 &= 0, \end{aligned}$$

while the unknowns $x_2, x_4$ satify the equations

$$\begin{aligned} ax_2 &+ bx_4 &= 0 \\ cx_2 &+ dx_4 &= 1. \end{aligned}$$

Solving these two systems one gets that $x_1 = \frac{d}{ad-bc}, x_2 = \frac{-b}{ad-bc}, x_3 = \frac{-c}{ad-bc}, x_4 = \frac{a}{ad-bc}$. So the elements of the matrix $B$ are uniquely defined by the elements of $A$. Now computing

$$BA = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we get that it is also equal to $I$.

*Second proof:* Reduce $B$ to the row reduced echelon matrix $B'$ by elementary row operations so that $B = E_1 \dots E_n B'$ for some elementary matrices $E_1, \dots, E_n$. Then the equality $AB = I$ implies that $B'$ is invertible from the left. Indeed,

$$(AE_1 \dots E_n)B' = I,$$

so the matrix $AE_1 \dots E_n$ is the left inverse of $B'$. Let us prove that a $2 \times 2$ row reduced echelon matrix that has a left inverse can not have zero rows. Otherwise, if

$$B' = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$$

has the bottom row zero, then for any matrix

$$A' = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the product

$$A'B' = \begin{pmatrix} ax & ay \\ cx & cy \end{pmatrix}$$

will have two proportional rows, so $A'B'$ can not be equal to the identity matrix. Since $B'$ does not have zero rows, it equals to the identity matrix. Hence, $B$ is the product of elementary matrices so it also has a right inverse $C = E_n^{-1} E_{n-1}^{-1} \dots E_1^{-1}$ such that $BC = I$. Now show that $A = C$. Indeed,

$$A = AI = A(BC) = (AB)C = IC = C.$$

Hence, BA=BC=I.

**pp 26-27**

**Exercise 2.** Let

$$A = \begin{pmatrix} 2 & 0 & i \\ 1 & -3 & -i \\ o & 1 & 1 \end{pmatrix}$$

Find a row-reduced echelon matrix $R$ which is row-equivalent to $A$ and an invertible $3 \times 3$ matrix $P$ such that $R = PA$.

**Solution:** $A$ is invertible, therefore we can take $R$ to be the identity matrix and $P = A^{-1}$:

$$A^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{30} - i\frac{1}{10} & \frac{1}{10} - \frac{3}{10}i \\ 0 & -\frac{3}{10} - \frac{1}{10}i & \frac{1}{10} - \frac{3}{10}i \\ -\frac{1}{3}i & \frac{1}{5} + \frac{1}{15}i & \frac{3}{5} + \frac{1}{5}i \end{pmatrix}$$

**Exercise 8.** Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Prove, using elementary row operations, that $A$ is invertible if and only if $(ad - bc) \neq 0$.

**Solution:** Start row reducing $A$. First, note that if $A$ is invertible or $ad - bc \neq 0$, then either $a$ or $c$ is not zero. Otherwise, $A$ would have a zero column, and for any $2 \times 2$ matrix $B$ the product $BA$ would also have a zero column so that $BA \neq I$. By interchanging rows we can assume that $a \neq 0$. Multiply the first row by $\frac{c}{a}$ and subtract it from the second one:

$$\begin{pmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{pmatrix}.$$

This matrix is invertible if and only if the second row is not zero, which means $d - \frac{bc}{a} \neq 0$. The latter is true if and only if $ad - bc \neq 0$.

**Exercise 9.** An $n \times n$ matrix is called *upper-triangular* if $A_{ij} = 0$ for $i > j$, that is, if every row below the main diagonal is $0$. Prove that an upper-triangular matrix is invertible if and only if every entry on its main diagonal is different from $0$.

**Solution:** Let $A$ be an upper triangular matrix. First, look at the bottom row of $A$. Its only (possibly) non-zero entry is the last one: $A_{nn}$. So if $A$ is invertible, then $A_{nn} \neq 0$. Otherwise, $A$ would have a zero row. By subtracting the multiples of the bottom row from the other rows we can eliminate all non-zero entries in the $n$-th column except for $A_{nn}$. Doing this will not change the other columns.

Now look at the $(n-1)$st row (it now also has only one possibly non-zero entry $A_{(n-1)(n-1)}$) and repeat the same procedure. We get that $A_{(n-1)(n-1)} \neq 0$. Repeating this $n$ times we prove that $A_{11}, \ldots, A_{nn} \neq 0$ and that $A$ is equivalent to the diagonal matrix with entries $A_{11}, \ldots, A_{nn}$ on the diagonal. The latter matrix is clearly invertible.

**Exercise 10.** Prove the following generalization of Exercise 6. If $A$ is an $m \times n$ matrix, $B$ is an $n \times m$ matrix and $m < n$, then $AB$ is not invertible.

**Solution** Let

$$A = \begin{pmatrix} a_{1,1} & \ldots & a_{1,m} \\ \vdots & & \vdots \\ a_{n,1} & \ldots & a_{n,m} \end{pmatrix}$$

And let

$$\tilde{A} = \begin{pmatrix} a_{1,1} & \dots & a_{1,m} & 0 & \dots & 0 \\ \vdots & & \vdots & 0 & \dots & 0 \\ a_{n,1} & \dots & a_{n,m} & 0 & \dots & 0 \end{pmatrix}$$

where the last $n-m$ entries are zero in each row. Notice that $\tilde{A}$ is not invertible, since any vector of the form $X = (\overbrace{0,\dots,0}^{n\ 0's}, \overbrace{x_1,\dots,x_{m-n}})$ is a solution of $\tilde{A}X = 0$. Also, let $\tilde{I}$ be the $m \times n$ matrix which has the first $n$ rows equal to the $n \times n$ identity matrix, and all other entries equal to zero, i.e.

$$\tilde{I} = \left( \frac{I_{n\times n}}{0_{(m-n)\times n}} \right)$$

Notice that $A = \tilde{A}\tilde{I}$. Now suppose that $AB$ is invertible, then there exists a matrix $P$ such that $(AB)P = I$ but then $\tilde{A}(\tilde{I}BP) = I$ which implies that $\tilde{A}$ is invertible! (*contradiction*). Therefore $AB$ is not invertible.

## pp. 33-34

**Exercise 4.** Let $V$ be the set of all pairs $(x, y)$ of real numbers, and let $F$ be the field of real numbers. Define

$$(x, y)\hat{+}(x_1, y_1) = (x + x_1, y + y_1)$$

$$c \cdot (x, y) = (cx, y)$$

Is $V$ with these operations, a vector space over the field of real numbers?

**Solution:** No. In a vector space we must have a unique vector $\hat{0}$ and the following equation must hold for any vector:

$$0 \cdot \alpha = \hat{0}$$

Notice that with the operations defined above we have

$$0 \cdot (0, 1) = (0, 1)$$

and

$$0 \cdot (0, 3) = (0, 3)$$

but this two products should be equal to the unique zero vector. Since $1 \neq 3$ we are done.

**Exercise 5** On $\mathbb{R}^n$, define two operations

$$\alpha \oplus \beta = \alpha - \beta$$

$$c \cdot \alpha = -c \cdot \alpha$$

Which of the axioms for a vector space are satisfied by $(\mathbb{R}^n, \oplus, \cdot)$ ?

**Solution:** Addition is **not** commutative, addition is **not** associative, there is a unique vector $0$ (namely, the usual $0$ vector), there is a unique inverse for every $\alpha$ (namely $\alpha$ itself), $1 \cdot \alpha \neq \alpha$, $c_1 \cdot (c_2 \cdot \alpha) = c_1 \cdot (-c_2\alpha) = -c_1(-c_2\alpha) = c_1c_2\alpha \neq -c_1c_2\alpha = (c_1c_2)\cdot\alpha$ , $c\cdot(\alpha\oplus\beta) = c\cdot\alpha\oplus c\cdot\beta = c(\beta-\alpha)$, $(c_1+c_2)\cdot\alpha = -(c_1+c_2)\cdot\alpha \neq (c_2-c_1)\cdot\alpha = c_1\cdot\alpha\oplus c_2\cdot\alpha$.

**Exercise 6.** Let $V$ be the set of all complex-valued functions $f$ on the real line such that (for all $t$ in $\mathbb{R}$)

$$f(-t) = \overline{f(t)}. \tag{1}$$

The bar denotes complex conjugation, i.e. $\overline{a+bi} = a - bi$. Show that $V$ is a vector space over the field of *real* numbers. Give an example of a function in $V$ that is not real-valued.

**Solution:** First, check that if functions $f, g$ satisfy equation $(1)$, then $f+g$ and $\lambda f$ for a real $\lambda$ also satisfy it. This is because complex conjugation commutes with opeartions of addition and multiplication by real numbers.

$$(f+g)(-t) = f(-t) + g(-t) = \overline{f(t)} + \overline{g(t)} = \overline{f(t)+g(t)} = \overline{(f+g)(t)},$$

$$(\lambda f)(-t) = \lambda f(-t) = \lambda\overline{f(t)} = \overline{\lambda f(t)}.$$

Hence, a subset $V$ of the real vector space of all functions from $\mathbb{R}$ to $\mathbb{C}$ is closed under addition and multiplication by real numbers. This means that $V$ is a subspace and satisfies all properties of a vector space.

An example of a non-real-valued function in $V$ is $f(t) = it$.