Homework Assignment 11

Section 7.3 pp. 249-250

Exercise 1. Let N_1 and N_2 be 3×3 nilpotent matrices over the field F. Prove that N_1 and N_2 are similar if and only if they have the same minimal polynomial.

Solution: If N_1 and N_2 are similar, they have the same minimal polynomial (cf. pg. 192). Conversely, suppose that N_1 and N_2 have the same minimal polynomial. The minimal polynomial must be x^k for some $1 \leq k \leq 3$. If $k = 1$ we get that the matrix is the zero matrix, so both N_1 and N_2 are the zero matrix. If $k = 2$ we get that the Jordan form for N_1 and N_2 is

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and thus, they are similar. If $k = 3$ then the Jordan form of both matrices is

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and thus, they are similar.

Exercise 3. If A is a complex 5×5 matrix with characteristic polynomial

$$f = (x-2)^3(x+7)^2$$

and minimal polynomial $p = (x-2)^2(x+7)$, what is the Jordan form for A?

Solution: The block matrix associated to the characteristic value 2 is a 3×3 matrix with 2's along the diagonal with an **elementary Jordan matrix** of size 2×2 (the multiplicity of 2 in the minimal polynomial) as the first block. i.e.:

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Analogously, the block matrix associated to the characteristic value -7 is a 2×2 matrix with -7's along the diagonal with an **elementary Jordan matrix** of size 1×1 as the first block, i.e.:

$$\begin{pmatrix} -7 & 0 \\ 0 & -7 \end{pmatrix}$$

Hence the Jordan form for A is:

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & 0 & -7 \end{pmatrix}$$

Exercise 4. How many possible Jordan forms are there for a 6×6 complex matrix with characteristic polynomial $(x+2)^4(x-1)^2$?

Solution: **ATTENTION!!!** There are 8 possibilities for the minimal polynomial, this implies that there are *at least* 8 different Jordan forms. But the minimal polynomial $(x+2)^2(x-1)$ may correspond to **TWO** different matrices, namely

$$\begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

this corresponds to the fact that $4 = 2+2$ but also $4 = 2+1+1$. Analogously the minimal polynomial $(x+2)^2(x-1)^2$ corresponds to **TWO** matrices. Therefore we have 10 different Jordan forms. Think of it in this way:
How many blocks corresponding to the eigenvalue -2 can we form?
This is equivalent to "In how many ways can we write 4 as a sum $a_1 + a_2 + ... + a_k$ with $a_i > 0$ and $a_1 \geq a_2 \geq \ldots \geq a_k$?" The answer is

$$\begin{aligned} 4 &= 4 \\ 4 &= 3+1 \\ 4 &= 2+2 \\ 4 &= 2+1+1 \\ 4 &= 1+1+1+1 \end{aligned}$$

i.e. in 5 different ways. Analogously, we can write 2 in only two different ways, namely $2 = 2$ and $2 = 1+1$. Therefore, multiplying we get 10 different Jordan forms.

Exercise 5. The differentiation operator on the space of polynomials of degree less than or equal to 3 is represented in the 'natural' ordered basis by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

What is the Jordan form of this matrix? (F a subfield of the complex numbers.)?

Solution The characteristic polynomial for this matrix is x^4 and the minimal polynomial is x^4, therefore the Jordan Form consists of only one block associated to the characteristic value 0 with its first **elementary Jordan block** of length 4. i.e. the Jordan form of the matrix is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Section 8.1 pp. 275-276

Exercise 2. Let V be a vector space over F. Show that the sum of two inner products on V is an inner product on V. Is the difference of two inner products an inner product? Show that a positive multiple of an inner product is an inner product.

Solution: Let f_1 and f_2 be two inner products, we will show that $f = f_1 + f_2$ satisfies all the axioms of an inner product.

(a) $f(a+b,c) = f_1(a+b,c) + f_2(a+b,c) = f_1(a,c) + f_1(b,c) + f_2(a,c) + f_2(b,c)$ the first equality is by definition, the second is because f_1 and f_2 are inner products. But reordering, the last term of the equality is equal to $f_1(a,c) + f_2(a,c) + f_1(b,c) + f_2(b,c) = f(a,c) + f(b,c)$, the last equality is again by definition.

(b) $f(ka,b) = f_1(ka,b) + f_2(ka,b) = kf_1(a,b) + kf_2(a,b) = k(f_1(a,b) + f_2(a,b) = kf(a,b)$

(c) $f(b,a) = f_1(b,a) + f_2(b,a) = \overline{f_1(a,b)} + \overline{f_2(a,b)} = \overline{f_1(a,b) + f_2(a,b)} = \overline{f(a,b)}$

(d) If $a \neq 0$, $f_1(a,a) > 0$ and $f_2(a,a) > 0$ therefore $f(a,a) = f_1(a,a) + f_2(a,a) > 0$

The difference of inner products is **NOT** an inner product in general: Let $f_1 = f_2$ and $f = f_1 - f_2$, and let $a \neq 0$, then $f_(a,a) = f_1(a,a) - f_2(a,a) = 0$ (which contradicts axiom d). The proof for the positive multiple of a scalar product is analogous to the prove for the sum.

Exercise 3. Describe all inner products on $\mathbb{R}^1$ and on $\mathbb{C}^1$

Solution: Let f be an inner product on $\mathbb{R}^1$, since f is linear on each variable we get:

$$f(r,s) = f(r \cdot 1, s \cdot 1) = rf(1, s \cdot 1) = rsf(1,1)$$

therefore the inner product of the vectors r and s is just the product of the real numbers r and s times $f(1,1)$. But we know that $f(1,1) > 0$. So we have as many inner products on R^1 as positive real numbers.
Let f be an inner product on $\mathbb{C}^1$, since f is linear on each variable we get:

$$f(r,s) = f(r \cdot 1, s \cdot 1) = rf(1, s \cdot 1) = r\overline{s}f(1,1)$$

therefore the inner product of the vectors r and s is just the product of the real numbers r and s times $f(1,1)$. But we know that $f(1,1) > 0$. So we have as many inner products on $\mathbb{C}^1$ as positive real numbers.

Exercise 5. Let $(|)$ be the standard inner product on $\mathbb{R}^2$.

(a) Let $\alpha = (1,2)$, $\beta = (-1,1)$. If γ is a vector such that $(\alpha|\gamma) = -1$ and $(\beta|\gamma) = 3$ find γ.

(b) Show that for any α in $\mathbb{R}^2$ we have $\alpha = (\alpha|e_1)e_1 + (\alpha|e_2)e_2$

Solution:

(a) We have to solve the system of equations

$$x_1 + 2x_2 = -1$$
$$-x_1 + x_2 = 3$$

Solving we get $\gamma = (-\frac{7}{3}, \frac{2}{3})$

(b) Writing α in terms of the standard basis we get $\alpha = a_1e_1 + a_2e_2$. On the right hand of the equation we get

$$(a_1e_1 + a_2e_2|e_1)e_1 + (a_1e_1 + a_2e_2|e_2)e_2 =$$
$$(a_1e_1|e_1)e_1 + (a_2e_2|e_1) + (a_1e_1|e_2)e_2 + (a_2e_2|e_2) =$$
$$a_1e_1 + a_2e_2$$

Section 8.2 pp. 288-289

Exercise 1. Consider $\mathbb{R}^4$ with the standard inner product. Let W be the subspace of $\mathbb{R}^4$ consisting of all vectors which are orthogonal to both $\alpha = (1, 0, -1, 1)$ and $\beta = (2, 3, -1, 2)$

Solution: We have to find the solution space for the system

$$\begin{aligned} x_1 - x_3 + x_4 &= 0 \\ 2x_1 + 3x_2 - x_3 + x_2 &= 0 \end{aligned}$$

Row reducing we get that the vectors $(1, -\frac{1}{3}, 1, 0)$ and $(-1, 0, 0, 1)$ form a basis for the solution space of the system.

Exercise 2. Apply the Gram-Schmidt process to the vectors $\beta_1 = (1, 0, 1)$, $\beta_2 = (1, 0, -1)$, $\beta_3 = (0, 3, 4)$, to obtain an orthonormal basis for $\mathbb{R}^3$ with the standard inner product.

Solution: Notice that the first two vectors are already orthogonal, therefore we only need to find the third vector. The basis we get is:

$$(1, 0, 1), (1, 0, -1), (0, 3, 0)$$

Exercise 9. Let V be the subspace of $\mathbb{R}[x]$ of polynomials of degree at most 3. Equip V with the inner product

$$(f|g) = \int_0^1 f(t)g(t)dt$$

(a) Find the orthogonal complement of the subspace of scalar polynomials.

(b) Apply the Gram-Schmidt process to the basis $\{1, x, x^2, x^3\}$

Solution:

(b) The basis we get is $\{1, x - \frac{1}{2}, \frac{1}{6} - x + x^2, -\frac{1}{20} + \frac{3}{5}x - \frac{3}{2}x^2 + x^3\}$

(a) Using (b), we obtain the following basis for the orthogonal complement of the subspace of scalar polynomials, i.e. the orthogonal complement of the subspace spanned by $\{1\}$: $\mathcal{B} = \{x - \frac{1}{2}, \frac{1}{6} - x + x^2, -\frac{1}{20} + \frac{3}{5}x - \frac{3}{2}x^2 + x^3\}$

Remark: When performing the Gram-Schmidt process it is helpful to have quick access to the inner products of the vectors of the basis. This can be achieved by writing the matrix $M = (m_{ij})$ where $m_{ij} = (e_i|e_j)$. In this way, if we write any two vectors α and β in terms of the basis $\{e_i\}$ we get the following: $(\alpha|\beta) = [\alpha]M[\beta]^T$. In the previous example we get the matrix

$$M = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{pmatrix}$$

so, for example, the inner product of $\alpha = 2 + 3x + x^2$ and $\beta = 8x^2 - x^3$ is equal to $[2, 3, 1, 0]M[0, 0, 8, -1]^T$.
With this notation, if $\mathcal{A} = \{1 = b_1, x = b_2, x^2 = b_3, x^3 = b_4\}$ the Gram-Schmidt process becomes:

$$\begin{aligned} a_1 &= b_1 \\ a_2 &= b_2 - \frac{b_2 M a_1^T}{a_1 M a_1^T} a_1 \\ a_3 &= b_3 - \frac{b_3 M a_1^T}{a_1 M a_1^T} a_1 - \frac{b_3 M a_2^T}{a_2 M a_2^T} a_2 \\ a_3 &= b_4 - \frac{b_4 M a_1^T}{a_1 M a_1^T} a_1 - \frac{b_4 M a_2^T}{a_2 M a_2^T} a_2 - \frac{b_4 M a_3^T}{a_3 M a_3^T} a_3 \end{aligned}$$