Homework assignment 10

Section 6.3 pp. 197-198

Exercise 1. Let V be a finite dimensional vector space. What is the minimal polynomial for the identity operator on V? What is the minimal polynomial for the zero operator?

Solution: The minimal polynomial for the identity operator is $p(x) = x - 1$. It is monic, of degree 1 and it annihilates the identity operator. The minimal polynomial for the zero operator is $p(x) = x$.

Exercise 3. Let A be the 4×4 real matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}$$

Show that the characteristic polynomial for A is $x^2(x-1)^2$ and that it is also the minimal polynomial.

Solution: Calculating $\det(xI - A)$ we get $x^2(x-1)^2$. The minimal polynomial should have the same degree 1 factors, i.e. x and $(x-1)$. Calculating the remaining possibilities we get: $A(A-I) \neq 0$, $A^2(A-I) \neq 0$, $A(A-1)^2 \neq 0$. Therefore the minimal polynomial is the characteristic polynomial.

Exercise 4. Is the matrix A of Exercise 3 similar over the field of complex numbers to a diagonal matrix?

Solution: One can easily check that the matrices A and $A - I$ have rank 3. Hence, A has exactly two eigenvectors: one with eigenvalue 0, and the other with eigenvalue 1. So A does not have a basis of eigenvectors, and thus is not similar to a diagonal matrix over the complex field.

Exercise 5. Let V be an n-dimensional vector space and let T be a linear operator on V. Suppose that there exists some positive integer k so that $T^k = 0$. Prove that $T^n = 0$.

Solution: If $T^k = 0$ then the minimal polynomial divides x^k, therefore the minimal polynomial must be x^s for some s between 1 and n (because the minimal polynomial has degree at most n), but then $T^s = 0$. Therefore we are reduced to the case when $T^k = 0$ with $k \leq n$. In that case $T^n = T^{n-k}T^k = T^{n-k} \cdot 0 = 0$.

Exercise 6. Find a 3×3 matrix for which the minimal polynomial is x^2.

Solution: Let

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Clearly $A^2 = 0$ but $A \neq 0$. Therefore the minimal polynomial is a monic polynomial which divides x^2 and is not x. i.e. it is x^2.

Section 6.4 pp. 205-206

Exercise 3. Let c be a characteristic value of T and let W be the space of characteristic vectors associated with the characteristic value c. What is the restriction operator T_W ?

Solution: Let w be any vector in W. Then w must satisfy $T(w) = cw$. But $T_W(w) = T(w)$. Therefore $T_W(w) = cw$, that is, $T = cI$.

Exercise 4. Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 2 & -2 & 2 \\ 2 & -3 & 2 \end{pmatrix}.$$

Is A similar to a triangular real matrix? If so, find such a triangular matrix.

Solution: Compute the characteristic polynomial of A. It is x^3. Therefore A is similar to a triangular matrix (the minimal polynomial divides x^3 and thus it is a product of linear factors). Find one eigenvector v_1 with eigenvalue 0 (solve the system $AX = 0$ we get $v_1 = (-1, 0, 1)$) . Now find a vector v_2 such that $Av_2 = v_1$ (solve the system $AX = v_1$, we get $(-1, -1, 0)$). Then find a vector v_3 such that $Av_3 = v_2$ (solve the system $AX = v_2$, we get $v_3 = (-\frac{3}{2}, -1, 0)$). In the basis $\mathcal{B} = \{v_1, v_2, v_3\}$ the operator whose matrix in the standard basis is A, will have an upper triangular matrix with zeroes on the diagonal, namely

$$[A]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Exercise 7. Let T be a linear operator on a finite-dimensional vector space over the field of complex numbers. Prove that T is diagonalizable if and only if T is annihilated by some polynomial over $\mathbb{C}$ which has distinct roots.

Solution: Suppose that T is diagonalizable, then by theorem 6 its minimal polynomial $p(x)$ factors as a product of polynomials of degree 1 with distinct roots, hence we are done.
Suppose now that T is annihilated by some polynomial $q(x)$ over $\mathbb{C}$ which has distinct roots, i.e. $q(x) = (x - c_1) \cdots (x - c_k)$ with $i \neq j \Rightarrow c_i \neq c_j$. Let $p(x)$ be the minimal polynomial of T. We know that $p(x) \mid q(x)$, i.e $q(x) = p(x)r(x)$. Since any polynomial over $\mathbb{C}$ factors as a product of linear factors, we only need to check that $p(x)$ has no multiple roots, to conclude that T is diagonalizable (again by theorem 6). Suppose that $p(x)$ has a multiple root a , i.e. $(x - a)^2 \mid p(x)$, then we must have $(x - a)^2 \mid q(x)$ (which contradicts our assumption on $q(x)$). Therefore $p(x)$ has no multiple roots and we are done.

Exercise 9. Let T be the indefinite integral operator

$$(Tf)(x) = \int_0^x f(t)dt$$

on the space of continuous functions on the interval $[0, 1]$. Is the space of polynomial functions invariant under T? The space of differentiable functions? The space of functions which vanish at $x = \frac{1}{2}$?

Solution: Using the fundamental theorem of calculus we get that the integral of a polynomial is a polynomial and the integral of a differentiable function is differentiable. Therefore the answer to the first two questions is yes. To answer the last question consider the function $f(x) = (x - \frac{1}{2})^2 = x^2 - x + \frac{1}{4}$. Note that $f(\frac{1}{2}) = 0$ and $(Tf)(\frac{1}{2}) = \int_0^{\frac{1}{2}} f(t)dt = \frac{37}{24}$, hence, the answer to the last question is no.