Problem	1	2	3	4	5	Bonus:	Total:
Points	**15**	**13**	**12**	**5**	**5**	**10**	**50+10**
Scores							

Mat 310 – Linear Algebra – Fall 2004

Name: ______________________ **Id. #:** **Lecture #:**

Test 1 (September 24 / 50 minutes)

There are 5 problems worth 50 points total and a bonus problem worth up to 10 points. Show all work. Always indicate carefully what you are doing in each step; otherwise it may not be possible to give you appropriate partial credit.

1. [15 points] Consider the homogeneous system of linear equations

$$\begin{aligned} x_1 + \ \ x_2 + 2x_3 - 2x_4 &= 0 \\ x_1 - 5x_2 - \ \ x_3 + 7x_4 &= 0 \\ x_1 - \ \ x_2 + \ x_3 + \ x_4 &= 0 \end{aligned}$$

(a) [3 points] Write down the matrices A, X, and O for which the system is in matrix form $AX = O$.

Solution:

$$A = \begin{pmatrix} 1 & 1 & 2 & -2 \\ 1 & -5 & -1 & 7 \\ 1 & -1 & 1 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad O = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(b) [6 points] Using the Gauss-Jordan algorithm, compute the row-reduced echelon matrix R which is row equivalent to A.

Solution:

$$A \xrightarrow{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 1 & 2 & -2 \\ 0 & -6 & -3 & 9 \\ 1 & -1 & 1 & 1 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{pmatrix} 1 & 1 & 2 & -2 \\ 0 & -6 & -3 & 9 \\ 0 & -2 & -1 & 3 \end{pmatrix} \xrightarrow{R_3 \to R_3 - \frac{1}{3} R_2} \begin{pmatrix} 1 & 1 & 2 & -2 \\ 0 & -6 & -3 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_2 \to -\frac{1}{6} R_2} \begin{pmatrix} 1 & 1 & 2 & -2 \\ 0 & 1 & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{pmatrix} 1 & 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(c) [6 points] Use (b) to find all solutions of the above system.

Solution:

$$x_1 = -\frac{3}{2} c_1 + \frac{1}{2} c_2, \quad x_2 = -\frac{1}{2} c_1 + \frac{3}{2} c_2, \quad x_3 = c_1, \quad x_4 = c_2,$$

where c_1, c_2 are any scalars.

2. [14 points] Let

$$A = \begin{bmatrix} 2 & -2 & 1 \\ 3 & -1 & 2 \\ 1 & -3 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

(a) [10 points] Apply row reduction to the augmented matrix of A and determine exactly for which triples (y_1, y_2, y_3) the system $AX = Y$ has a solution.

Solution:

$$\begin{pmatrix} 2 & -2 & 1 & y_1 \\ 3 & -1 & 2 & y_2 \\ 1 & -3 & 0 & y_3 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix} 2 & -2 & 1 & y_1 \\ -1 & 3 & 0 & y_2 - 2y_1 \\ 1 & -3 & 0 & y_3 \end{pmatrix} \xrightarrow{R_3 \to R_2 + R_3} \begin{pmatrix} 2 & -2 & 1 & y_1 \\ -1 & 3 & 0 & y_2 - 2y_1 \\ 0 & 0 & 0 & y_3 + y_2 - 2y_1 \end{pmatrix}$$

$$\xrightarrow{R_1 \to R_1 + 2R_2} \begin{pmatrix} 0 & 4 & 1 & y_2 - y_1 \\ -1 & 3 & 0 & y_2 - 2y_1 \\ 0 & 0 & 0 & y_3 + y_2 - 2y_1 \end{pmatrix} \xrightarrow{R_2 \to -R_2} \begin{pmatrix} 0 & 4 & 1 & y_2 - y_1 \\ 1 & -3 & 0 & -y_2 + 2y_1 \\ 0 & 0 & 0 & y_3 + y_2 - 2y_1 \end{pmatrix}.$$

The system $AX = Y$ has solutions iff $y_3 + y_2 - 2y_1 = 0$.

(b) [3 points] Write down an explicit right hand side column Y for which the system $AX = Y$ has no solution.

Solution:

$$Y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

3. [12 points] Consider

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}.$$

(a)[8 points] By any method, argue that A is invertible and compute A^{-1}.

Solution:

$$\begin{pmatrix} 2 & 3 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \to R_2 - \frac{3}{2}R_1} \begin{pmatrix} 2 & 3 & 1 & 0 \\ 0 & -\frac{1}{2} & -\frac{3}{2} & 1 \end{pmatrix} \xrightarrow{R_2 \to -2R_2} \begin{pmatrix} 2 & 3 & 1 & 0 \\ 0 & 1 & -3 & 2 \end{pmatrix} \xrightarrow{R_2 \to -2R_2} \begin{pmatrix} 2 & 3 & 1 & 0 \\ 0 & 1 & -3 & 2 \end{pmatrix}$$

$$\xrightarrow{R_1 \to R_1 - 3R_2} \begin{pmatrix} 2 & 0 & 8 & -6 \\ 0 & 1 & -3 & 2 \end{pmatrix} \xrightarrow{R_1 \to \frac{1}{2}R_1} \begin{pmatrix} 1 & 0 & 4 & -3 \\ 0 & 1 & -3 & 2 \end{pmatrix}.$$

(b)[4 points] Write A as a product of elementary matrices.

Solution:

$$A = \begin{pmatrix} 1 & 0 \\ -\frac{3}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$$

4. [5 points] Find an example of 2×2 matrices A, B for which it is *not* true that $(A+B)^2 = A^2 + 2AB + B^2$. [Can you give a condition for A and B, so the last matrix formula would hold? You'll get 5 extra points for the right answer.]

Solution: Note that $(A+B)^2 = A^2 + AB + BA + B^2$ for all matrices A and B. Hence, $(A+B)^2 = A^2 + 2AB + B^2$ if and only if $AB = BA$. Take matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $AB \neq 0$, but $BA = 0$, so $AB \neq BA$.

5. [5 points] Let A be an $n \times n$ matrix. Show that A is invertible if and only if A^2 is invertible.

Solution: If A has the inverse B so that $AB = BA = I$, then A^2 has the inverse B^2 because

$$A^2B^2 = A\underbrace{AB}_{I}B = AB = I,$$

and the same for B^2A^2.

If A^2 has the inverse C so that $A^2C = CA^2 = I$, then A has the left inverse AC (since $A(AC) = A^2C = I$) and the right inverse CA (since $(CA)A = CA^2 = I$). And if a matrix have both right and left inverses, then these inverses coincide:

$$(AC) = I(AC) = ((CA)A)AC = (CA)(A(AC)) = (CA)I = (CA).$$

Bonus Problem [up to 10 points] Consider the equation $X^2 = -I$ for 2×2 matrices. There are solutions with real coefficient. Discover as many as you can. Can you find infinitely many? Give some more additional solutions with complex coefficients. [You might also want to observe that if P is invertible 2×2, then if X solves, so does PXP^{-1}. Why?]

Solution: Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then

$$A^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence, $b(a+d) = c(a+d) = 0, a^2 + bc = d^2 + bc = -1$. If $a + d \neq 0$, then $b, c = 0$ and $a^2 = d^2 = -1$ so a, d can not be real in this case. If $a + d = 0$, we get that $a = -d$ and $bc = -1 - a^2$, so all solutions of the equation $A^2 = -I$ have form

$$\begin{pmatrix} a & b \\ \frac{-1-a^2}{b} & -a \end{pmatrix},$$

where a, b are real numbers.

Additional complex solutions are, for instance,

$$\begin{pmatrix} \pm i & 0 \\ 0 & \pm i \end{pmatrix}.$$

Note that if A is a solution, then $(PAP^{-1})^2 = PAP^{-1}PAP^{-1} = PA^2P^{-1} = -PIP^{-1} = -I$, so PAP^{-1} is also a solution. Actually, if one solution A with real coefficients is known, then all real (or complex) solutions have the form PAP^{-1} for some matrix P with real (or complex) coefficients.