First Midterm Exam MAT 310, Spring 2005

Name: **ID#:** **Rec:**

problem	1	2	3	4	5	Total
possible	20	20	20	20	20	100
score						

Directions: There are 5 problems on five pages in this exam. Make sure that you have them all. Do all of your work in this exam booklet, and cross out any work that the grader should ignore. You may use the backs of pages, but indicate what is where if you expect someone to look at it. **Books, calculators, extra papers, and discussions with friends are not permitted.** Feel free to confer with the psychic friends network if you can do so silently. However, I don't think Dionne Warwick knows much linear algebra.

1. *(20 points)* Let $\mathbb{V}$ be the (real) vector space of all functions f from $\mathbb{R}$ into $\mathbb{R}$.

a.) Is $\mathbb{W} = \{f \mid f(\pi^2) = f(2)\}$ a subspace of $\mathbb{V}$? Prove or give a reason why not.

b.) Is $\mathbb{W} = \{f \mid [f(\pi)]^2 = f(2)\}$ a subspace of $\mathbb{V}$? Prove or give a reason why not.

2. *(20 points)* Let T be the transformation from $\mathbb{C}^3$ to $\mathbb{C}^3$ corresponding to the matrix

$$\begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 1 \\ i & 1 & 0 \end{pmatrix}$$

a) What is the rank of T? Write a basis for the image of T.

b) What is the nullity of T? Write a basis for the null space of T (also called the kernel of T).

c) Is T invertible? Justify your answer.

3.*(20 points)* Let $\mathbb{V}$ be a vector space over $\mathbb{F}$, and let W_1 and W_2 be subspaces of $\mathbb{V}$. Suppose also that

- $W_1 + W_2 = \mathbb{V}$
- $W_1 \cap W_2 = \{0\}$

Prove that for any vector $\alpha \in \mathbb{V}$, α can be written *uniquely* as $\alpha = \alpha_1 + \alpha_2$, where $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$.

4. *(20 points)* Let $\mathcal{P}_3$ be the vector space of polynomials of degree at most 3, and let

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_2(x) &= 2x^2 - 1 \\ T_3(x) &= 4x^3 - 3x \end{aligned}$$

a) Show that $\mathcal{B} = \{T_0, T_1, T_2, T_3\}$ is a basis for $\mathcal{P}_3$.

b) What are the coordinates of the polynomial $p(x) = x^3 - x^2$ in the ordered basis $\mathcal{B} = \{T_0, T_1, T_2, T_3\}$?

5. *(20 points)* Let $\mathbb{V}$ and $\mathbb{W}$ be finite dimensional vector spaces over a field $\mathbb{F}$. Prove that $\mathbb{V}$ is isomorphic to $\mathbb{W}$ if and only if $\dim \mathbb{V} = \dim \mathbb{W}$. (Hint: Considering the bases $\mathcal{B}_{\mathbb{V}} = \{\alpha_1, \ldots, \alpha_n\}$ and $\mathcal{B}_{\mathbb{W}} = \{\beta_1, \ldots, \beta_m\}$ may be useful.)