Solutions to First Midterm Exam MAT 310, Spring 2005

Name: **ID#:** **Rec:**

problem	1	2	3	4	5	Total
possible	20	20	20	20	20	100
score						

Directions: There are 5 problems on five pages in this exam. Make sure that you have them all. Do all of your work in this exam booklet, and cross out any work that the grader should ignore. You may use the backs of pages, but indicate what is where if you expect someone to look at it. **Books, calculators, extra papers, and discussions with friends are not permitted.** Feel free to confer with the psychic friends network if you can do so silently. However, I don't think Dionne Warwick knows much linear algebra.

1. *(20 points)* Let $\mathbb{V}$ be the (real) vector space of all functions f from $\mathbb{R}$ into $\mathbb{R}$.

a.) Is $\mathbb{W} = \{f \mid f(\pi^2) = f(2)\}$ a subspace of $\mathbb{V}$? Prove or give a reason why not.

Solution: Yes, it is a subspace. To prove this, let f and g be functions in $\mathbb{W}$ and let c be any scalar. We need to see that the function $cf + g$ is also in $\mathbb{W}$. But

$$(cf+g)(\pi^2) = cf(\pi^2) + g(\pi^2) = cf(2) + g(2) = (cf+g)(2)$$

b.) Is $\mathbb{W} = \{f \mid [f(\pi)]^2 = f(2)\}$ a subspace of $\mathbb{V}$? Prove or give a reason why not.

Solution: No, it is not, since

$$[(cf)(\pi)]^2 = c^2[f(\pi)]^2 = c^2 f(2) \neq cf(2)$$

unless $c = 0$ or $c = 1$ (or $f(2) = 0$).

2. *(20 points)* Let T be the transformation from $\mathbb{C}^3$ to $\mathbb{C}^3$ corresponding to the matrix

$$\begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 1 \\ i & 1 & 0 \end{pmatrix}$$

a) What is the rank of T? Write a basis for the image of T.

Solution: Row reducing gives us

$$\begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 1 \\ i & 1 & 0 \end{pmatrix} \xrightarrow{R_3 \to R_3 - iR_1} \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

which is in reduced echelon form. Thus, $\{(1,0,i),(0,1,1)\}$ is a basis for the image of T, and the rank of T is 2.

b) What is the nullity of T? Write a basis for the null space of T (also called the kernel of T).

Solution: Since $\dim \mathbb{C}^3 = rank(T) + nullity(T)$ and $\dim \mathbb{C}^3 = 3$, $rank(T) = 2$, the nullity of T must be 1.

To find a basis for the kernel of T, we must find one nonzero vector v such that $T(v) = 0$ (we only need one since it is a one dimensional vector space). Notice that $T(i, 1, -1) = (0, 0, 0)$, so $\{(-i, -1, 1)\}$ is a basis for the kernel of T.

c) Is T invertible? Justify your answer.

Solution: No. Since T has a nontrivial kernel, it is not one-to-one. Alternatively, since T is not of full rank, it is not onto. Either shows it is not invertible.

3.*(20 points)* Let $\mathbb{V}$ be a vector space over $\mathbb{F}$, and let W_1 and W_2 be subspaces of $\mathbb{V}$. Suppose also that

- $W_1 + W_2 = \mathbb{V}$
- $W_1 \cap W_2 = \{0\}$

Prove that for any vector $\alpha \in \mathbb{V}$, α can be written *uniquely* as $\alpha = \alpha_1 + \alpha_2$, where $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$.

Solution: Since $W_1 + W_2 = V$, every vector $\alpha \in V$ can be represented as

$$\alpha = \alpha_1 + \alpha_2$$

for some $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$.

We must then show that this representation is unique. Suppose that there are two other vectors $\beta_1 \in W_1$ and $\beta_2 \in W_2$ such that

$$\alpha = \beta_1 + \beta_2.$$

Then

$$\alpha - \alpha = 0 = (\alpha_1 + \alpha_2) - (\beta_1 + \beta_2)$$

so

$$\alpha_1 - \beta_1 = \beta_2 - \alpha_2.$$

Call the above vector γ. Then since α_1 and β_1 are both in W_1, their difference γ must also be in W_1. Similarly, γ must be in W_2, and so

$$\gamma \in W_1 \cap W_2.$$

Therefore $\gamma = 0$, that is, $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$.

4. *(20 points)* Let $\mathcal{P}_3$ be the vector space of polynomials of degree at most 3, and let

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_2(x) &= 2x^2 - 1 \\ T_3(x) &= 4x^3 - 3x \end{aligned}$$

a) Show that $\mathcal{B} = \{T_0, T_1, T_2, T_3\}$ is a basis for $\mathcal{P}_3$.

Solution: Either we can show that $\mathcal{B}$ consists of linearly independent vectors which span$\mathcal{P}_3$, or we can exhibit an isomorphism between the standard basis for $\mathcal{P}_3$ and this one. Exhibiting the isomorphism is easier, since we can just read it off from the description of $\mathcal{B}$. It is the linear map $S : \mathcal{P}_3 \to \mathcal{P}_3$ for which $S(1) = 1$, $S(x) = x$, $S(x^2) = 2x^2 - 1$ and $S(x^3) = 4x^3 - 3x$. This is the map corresponding to the matrix

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

which has rank 4. Since we know any linear map of full rank must be an isomorphism, we are done.

If you don't like that way, we can show that $\{T_0, T_1, T_2, T_3\}$ is a linearly independent set. Suppose that

$$a_0T_0 + a_1T_1 + a_2T_2 + a_3T_3 = 0$$

We want to conclude that a_0,a_1,a_2, and a_3 are all 0. Expanding the definitions of the T_i and collecting powers of x, we get

$$a_3x^3 + a_2x^2 + (a_1 - 3a_3)x + (a_2 - a_0) = 0$$

for all values of x. Hence, $a_3 = 0$ and $a_2 = 0$. Since $a_1 - 3a_3 = 0$ and $a_3 = 0$, we must also have $a_1 = 0$, and finally, since $a_2 = 0$ we must have $a_0 = 0$.

Since we have a linearly independent set of 4 vectors, the span is of dimension 4, so it is a basis.

If you prefer, you can instead (or as well) check that the basis spans. To do this, we must show that any cubic polynomial of the form

$$p(x) = ax^3 + bx^2 + cx + d$$

can be written as a combination of the T_i. The desired polynomial is

$$p(x) = aT_3(x) + bT_2(x) + (3a + c)T_1(x) + (b + d)T_0(x).$$

So $\mathcal{B}$ spans.

Since we have a set of 4 vectors which span a 4-dimensional space, they must be linearly independent, and hence $\mathcal{B}$ is a basis.

b) What are the coordinates of the polynomial $p(x) = x^3 - x^2$ in the ordered basis $\mathcal{B} = \{T_0, T_1, T_2, T_3\}$?

Solution: We need to solve

$$a_0T_0(x) + a_1T_1(x) + a_2T_2(x) + a_3T_3(x) = x^3 - x^2$$

That is,

$$a_0 + a_1x + a_2(2x^2 - 1) + a_3(4x^3 - 3x) = x^3 - x^2.$$

Equating the coefficients of the powers of x gives us:

$$\begin{array}{ccc} rcl4a_3 & = & 1 \\ 2a_2 & = & -1 \\ a_1 - 3a_3 & = & 0 \\ a_0 - a_2 & = & 0 \end{array}$$

We can just read off that $a_3 = 1/4$ and $a_2 = -1/2$. Since $a_1 = 3a_3$, we must have $a_1 = 3/4$, and since $a_0 = a_2$, we have $a_0 = -1/2$. Thus, the coordinates of $p(x)$ in the basis $\mathcal{B}$ are

$$\left[-\frac{1}{2}, \frac{3}{4}, -\frac{1}{2}, \frac{1}{4}\right].$$

5. *(20 points)* Let $\mathbb{V}$ and $\mathbb{W}$ be finite dimensional vector spaces over a field $\mathbb{F}$. Prove that $\mathbb{V}$ is isomorphic to $\mathbb{W}$ if and only if $\dim \mathbb{V} = \dim \mathbb{W}$. (Hint: Considering the bases $\mathcal{B}_{\mathbb{V}} = \{\alpha_1, \ldots, \alpha_n\}$ and $\mathcal{B}_{\mathbb{W}} = \{\beta_1, \ldots, \beta_m\}$ may be useful.)

Solution: This solution is currently omitted so that you can do this yourself, for extra credit and for the writing requirement.