In a previous class, we saw that the positive reals $\mathbb{R}^+$ is a vector space over a field $\mathbb{F}$ (where $\mathbb{F}$ is $\mathbb{R}$ or a subfield of $\mathbb{R}$ such as the rationals $\mathbb{Q}$), provided that we use multiplication in $\mathbb{R}^+$ for vector addition, and raise our vector to the power c for scalar multiplication by c. That is, for $\alpha, \beta \in \mathbb{R}^+$ and $c \in \mathbb{F}$, we set

$$\alpha \boxplus \beta = \alpha\beta \qquad \text{and} \qquad c \boxdot \alpha = \alpha^c$$

In this note, we want to demonstrate that this vector space is the same as the vector space of $\mathbb{R}$ over $\mathbb{F}$ (with ordinary addition and scalar multiplication). We'll continue to use $\boxplus$ to represent vector addition in $\mathbb{R}^+$, and $\boxdot$ to represent scalar multiplication.

First, notice that if our field $\mathbb{F}$ is the real numbers $\mathbb{R}$, our vector space $\mathbb{R}^+$ is a one-dimensional vector space, and so *must* be isomorphic to $\mathbb{R}$ with ordinary addition (since they are both finite dimensional vector spaces over the same field and have the same dimension.)

To see that, let's find a basis for $\mathbb{R}^+$. Any positive real number other than 1 will do, so let's use 2. To show that $\{2\}$ is a basis, we must prove that every other element of $\mathbb{R}^+$ is a linear combination of 2. That means that given any $x \in \mathbb{R}^+$, we can find a scalar $c \in \mathbb{F}$ for which

$$x = c \boxdot 2 \quad \text{or, equivalently,} \quad x = 2^c.$$

But certainly $c = \log_2 x$ is the desired solution. Since we found one vector which spans, $\mathbb{R}^+$ is a one-dimensional vector space over the field $\mathbb{R}$.

If we use a subfield of $\mathbb{R}$ for $\mathbb{F}$, then depending on which x we choose, $\log_2 x$ may or may not be an element of $\mathbb{F}$, and so we will need more than one element in our basis for this other vector space. For example, if $\mathbb{F} = \mathbb{Q}$, we will need an infinite basis.

The above discussion does more than show us the dimension of our vector space. It also gives us an isomorphism between $\mathbb{R}^+$ and $\mathbb{R}$. To see this, we must verify that the map $\log_2$ is one-to-one, onto, and linear.

- If $\log_2 x = \log_2 y$, then $2^{\log_2 x} = 2^{\log_2 y}$, and so $x = y$.
- For any $y \in \mathbb{R}$, we must find an $x \in \mathbb{R}^+$ so that $\log_2 x = y$. But $x = 2^y$ works for us here, so $\log_2$ is onto.
- At first, you might worry that the logarithm isn't a linear map. But keep in mind that it *is* linear with our unusual definition for addition and scalar multiplication. We have to show that for any $\alpha, \beta \in \mathbb{R}^+$ and for any $c \in \mathbb{F}$, we have
$$\log_2 \left((c \boxdot \alpha) \boxplus \beta\right) = c \log_2 \alpha + \log_2 \beta,$$
(where on the right side we are using ordinary addition and multiplication).

To see that, we just expand:

$$\begin{aligned}\log_2(c \boxdot \alpha \boxplus \beta) &= \log_2(\alpha^c\beta,) \\ &= \log_2(\alpha^c) + \log_2(\beta) \\ &= c\log_2\alpha + \log_2(\beta)\end{aligned}$$

as desired.

Since we have an isomorphism between $\mathbb{R}^+$ and $\mathbb{R}$, we can think of these two vector spaces as “the same”.