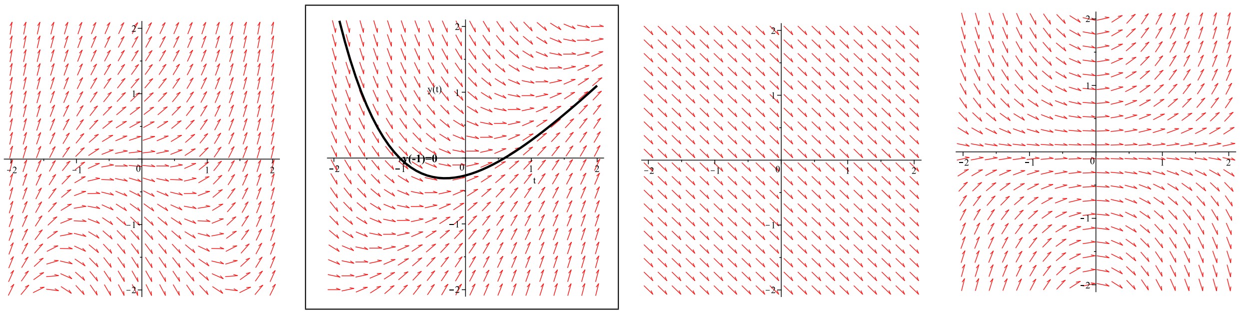
MATH 308 Solutions to Midterm 1

1. Consider the differential equation

$$\frac{dy}{dt} = t - y$$

3 pts. (a) Circle the direction field below which corresponds to this equation.



3 pts. (b) Based on your answer above, if $y(-1) = 0$, circle the statement below which is true.

A. $y(2) < -1$ B. $y(2) \approx 0$ C. $y(2) \approx 1$ D. $y(2) > 2$

Solution: You can see from the solution on the graph above that the solution passing through $(-1, 0)$ passes close to $(2, 1)$, so the correct choice is **C**.

Alternatively, you could use Euler's method with stepsize 1: $y(-1) = 0$, and so $y(0) \approx y(-1) + (-1 - 0) = -1$, $y(1) \approx y(0) + (0 - (-1)) = -1 + 1 = 0$, and $y(2) \approx y(1) + (1 - 0) = 1$.

Or, you can do part c below, and plug in $t = 2$ to get $1 + 2e^{-3}$, which is about 1.09957.

10 pts. (c) Find a function $y(t)$ such that $\frac{dy}{dt} = t - y$ and $y(-1) = 0$.

Solution: Rewrite the equation as $y' + y = t$, and then use the integrating factor $e^{\int 1\,dt} = e^t$ to obtain

$$(y' + y)e^t = te^t$$

$$\frac{d}{dt}(ye^t) = te^t$$

$$ye^t = \int te^t\,dt = te^t - e^t + c, \quad \text{(using integration by parts)}$$

Dividing both sides by e^t gives us $y = t - 1 + ce^{-t}$.

Note that you could also have found the general solution by observing that the general solution to $y' + y = 0$ is ce^{-t}, and (observed from the picture above) $y = t - 1$ is a particular solution to the given equation. Thus, the general solution must be $y(t) = ce^{-t} + t - 1$.

Since the desired solution has $y = 0$ when $t = -1$, we have $0 = -2 + ce$, or $c = 2/e$. Thus our solution is

$$\boxed{y(t) = t - 1 + 2e^{-t-1}}$$

8 pts. 2. The matrix $A = \begin{pmatrix} 11 & 3 & -3 \\ 8 & 6 & -8 \\ 5 & -5 & 3 \end{pmatrix}$ has eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

Find a matrix P such that $C = PAP^{-1}$ is a diagonal matrix; your answer should list both C and P.

Solution: Since we have three eigenvalues and the image has dimension 3, the eigenvalues span the image. Thus, in the coordinate system with eigenvalues as a basis, the transformation will be represented as a diagonal matrix.

The matrix P^{-1} sends the basis $\{e_1, e_2, e_3\}$ to the eigenvalues. Thus

$$P^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad \text{and so} \qquad \boxed{P = \tfrac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}}$$

To find C, we can just calculate the eigenvalues of A:

Observe that $A\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 8 \end{pmatrix}$, so the eigenvalue is 8. Also, the eigenvalue for $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is -2, and the other eigenvalue is 14. Consequently, $\boxed{C = \begin{pmatrix} 8 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 14 \end{pmatrix}}$

Of course, you could also find C by multiplication:

$$C = PAP^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 11 & 3 & -3 \\ 8 & 6 & -8 \\ 5 & -5 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 4 & -4 & 4 \\ 1 & -1 & -1 \\ 7 & 7 & -7 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 14 \end{pmatrix}$$

3. Consider the set V consisting of cubic polynomials $p(x)$ defined on $[-1, 1]$ for which $p(-1) = p(1)$.

5 pts. (a) Show that this is a subspace of the vector space $\mathcal{P}_3[-1, 1]$ consisting of cubic polynomials defined on $[-1, 1]$.

Solution: We need to confirm that if $p(x)$ and $q(x)$ are both in V, then so is $p(x) + q(x)$ and if c is any scalar, then $cp(x)$ is also in V.

Since $p \in V$, we know $p(-1) = p(1)$, and since $q \in V$ we have $q(-1) = q(1)$. But then of course $p(-1) + q(-1) = p(1) + q(1)$, so $p(x) + q(x)$ is in V. Similarly, $cp(1) = cp(-1)$, so $cp(x) \in V$.

8 pts. (b) Find a basis for V. (You need to show it is a basis).

Solution: Note any cubic polynomial is of the form $p(x) = ax^3 + bx^2 + cx + d$. Since $p(1) = a + b + c + d$ and $p(-1) = -a + b - c + d$, for $p(x)$ to be in V we must have $a + c = 0$. Thus, every polynomial in V is of the form $ax^3 + bx^2 - ax + d$, and a basis for V is $\boxed{\{1, x^2, x^3 - x\}}$.

Another, equivalent way of doing this is as follows:

The dimension of V is at most 3 (since $\mathcal{P}_3$ has dimension 4, and the polynomial x is not in V). This means if we find three linearly independent polynomials, these *must* form a basis.

Note that constant functions are in V, as is x^2. Finally, the cubic $p(x) = x^3 - x$ is also in V, since $p(-1) = 0 = p(1)$.

These three polynomials are linearly independent, since if $a(x^3 - x) + bx^2 + d \equiv 0$, we must have $a = b = d = 0$.

Thus $\{1, x^2, x^3 - x\}$ is a basis for V.

10 pts. (c) Put the inner product

$$\langle p(x), q(x) \rangle = \int_{-1}^{1} p(x)q(x)dx$$

on V, and find a non-zero polynomial $r(x)$ in V which is orthogonal to the span of the set of polynomials $\{1, x^2\}$. (You must show your answer is indeed orthogonal to this space).

Solution: Let $p(x) = x^3 - x$. If we subtract off the projections of $p(x)$ onto 1 and x^2, we will obtain the desired fuction. That is,

$$r(x) = p(x) - \frac{\langle p(x), 1 \rangle}{\langle 1, 1 \rangle} - \frac{\langle p(x), x^2 \rangle}{\langle x^2, x^2 \rangle} x^2.$$

But observe that

$$\begin{aligned}
\langle p(x), 1 \rangle &= \int_{-1}^{1} x^3 - x\, dx &= \frac{x^4}{4} - \frac{x^2}{2}\Big|_{-1}^{1} &= 0 \\
\left\langle p(x), x^2 \right\rangle &= \int_{-1}^{1} x^5 - x^3\, dx &= \frac{x^6}{6} - \frac{x^4}{4}\Big|_{-1}^{1} &= 0
\end{aligned}$$

This means that $x^3 - x$ is already orthogonal to the given polynomials, so life is good. We can take $\boxed{r(x) = x^3 - x}$.

4. Let $T(x, y, z)$ be the linear transformation such that

$$T(1,0,0) = (1,1,1), \qquad T(1,0,1) = (1,1,2), \qquad T(1,1,1) = (3,3,5)$$

5 pts. (a) Write the matrix corresponding to T.

Solution: We need to determine $T(0,1,0)$ and $T(0,0,1)$. But because of linearity,

$$T(0,0,1) = T(1,0,1) - T(1,0,0) = (1,1,2) - (1,1,1) = (0,0,1).$$

and

$$T(0,1,0) = T(1,1,1) - T(1,0,1) = (3,3,5) - (1,1,2) = (2,2,3).$$

Thus the matrix for T is $\begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{pmatrix}$.

5 pts. (b) Give a basis for the image of T.

Solution: The image is the span of the columns. Since $(2,2,3) = 2(1,1,1) + (0,0,1)$, we only need two of the columns in our basis. Thus $\boxed{\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}}$ will do the job, (as will any two of the columns of T).

5 pts. (c) Give a basis for the kernel of T.

Solution: Since the image has dimension 2, the kernel has dimension 1. Thus, we need to find any nonzero vector v for which $T(v) = 0$.
Note that $T(x, y, z) = (x + 2y, x + 2y, x + 3y + z)$, so we must have

$$x + 2y = 0, \quad \text{and} \quad x + 3y + z = 0.$$

If (x, y, z) is in the kernel, we must have $x = -2y$ and $y = -z$. Consequently, any vector in the kernel is in the $\boxed{\text{span of } \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}}$.

$\boxed{\text{15 pts.}}$ 5. Find all real eigenvalues and corresponding eigenvectors for the linear transformation corresponding to the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & -1 \\ 4 & 0 & 3 \end{pmatrix}$$

Solution: To find the eigenvalues, we compute the characteristic polynomial:

$$\det \begin{pmatrix} 1-\lambda & 0 & 0 \\ -2 & 2-\lambda & -1 \\ 4 & 0 & 3-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda)(3-\lambda)$$

which has roots $\lambda \in \{1, 2, 3\}$. Thus, the $\boxed{\text{eigenvalues are } 1, 2, \text{ and } 3}$.

The eigenvector for 1 satisfies

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & -1 \\ 4 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \text{that is} \quad \begin{pmatrix} 0 & 0 & 0 \\ -2 & 1 & -1 \\ 4 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Equivalently, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, so $\boxed{\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \text{ is an eigenvector for } 1}$.

To get an eigenvector for 2, observe that

$$\begin{pmatrix} 1-2 & 0 & 0 \\ -2 & 2-2 & -1 \\ 4 & 0 & 3-2 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ -2 & 0 & -1 \\ 4 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ so } \boxed{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ is an eigenvector for } 2}.$$

Finally,

$$\begin{pmatrix} 1-3 & 0 & 0 \\ -2 & 2-3 & -1 \\ 4 & 0 & 3-3 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ -2 & -1 & -1 \\ 4 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \boxed{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \text{ is an eigenvector for } 3}.$$